PRINCIPLES and METHODS

...OF...

Arithmetic Teaching

By WILLIAM B. CHRISWELL, Ph. B.

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PREFACE

The following pages have been prepared for the use of students in the Potsdam State Normal School.

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W. B. C.

PART ONE

GENERAL PRINCIPLES.

It has been well said that we should be loyal to principles but the slave to no man's devices. Only a few of the general principles especially applicable to Arithmetic teaching can be treated in this course, and these but briefly. For fuller treatment the student is referred to Young, The Teaching of Mathematics; Smith, The Teaching of Elementary Mathematics; Suzzallo, The Teaching of Primary Arithmetic; McMurry, Special Method in Arithmetic; McLellan and Dewey, The Psychology of Number; Brooks, Philosophy of Arithmetic; De Garmo, Interest and Education; and general works on Psychology and Education.

I. REASONS FOR THE STUDY OF METHOD.

It is not so much whether a teacher is a good teacher as whether she is a progressive teacher. The good teacher may deteriorate, the progressive teacher never. Proper method is as essential to the teacher as to the singer or the painter. Without some knowledge of the technique of the profession, little progress can be made.

Teachers may be classified as follows:

I. UNINSPIRED.

1. The unskilled are those who have neither natural ability for teaching nor knowledge of technique. The sooner such are eliminated form the profession the better.

2. The skilled are those who have little natural talent for teaching, but who have through the study of method mastered the technique of the profession. These are good teachers but lacking in inspiration.

II. INSPIRED.

1. The born teacher is full of enthusiam and inspiration, but lacking in knowledge of technique. Her methods are often faulty, but this is more than counterbalanced by the inspiration imparted to others.

2. The artists are the born teachers who have become masters of technique. They are the Michael Angelos, the Rembrandts, the Millets of the teaching profession.

The purpose of the study of method is to transform into skilled teachers those who would otherwise be unskilled, and into artists those who are born teachers. To quote from Froebel:

"Step by step lift bad to good;
Without halting, without rest,
Lifting better up to best."

See Young, Chap. I, VIII, and pp. 254 and 256; Page, Theory and Practice of Teaching, pp. 19-29; Suzzallo, Chap. III; Young, Teaching of Mathematics in Prussia, Chap. IV.

II. AIMS OF ARITHMETIC TEACHING.

Various aims of arithmetic teaching have been given, the chief being intellectual training, and utilitarian purposes. The reasoning connected with the solution of problems fulfills the intellectual aim. For practical use the essentials are knowledge of number facts; fundamental operations in integers, fractions and decimals; practical problems including percentage; and absolute accuracy and a resasonable degree of rapidity. Accuracy is the sine qua non.. "Careless facility", says Professor Frank Hall, "is not merely useless, it is positively harmful." See Smith, Chap. II; Young, Chap. II; and pp. 203 to 209, 214 to 216; Suzzallo, Chap II; Horne, Philosophy of Education, pp. 115, 116; Spencer, Education, pp. 29-31; Brown and Coffman, How to teach Arithmetic, Chap. III.

III. FUNCTION AND EXTENT OF OBJECTIVE TEACHING.

Clear perception is essential as a basis for future knowledge. At first new ideas must be presented directly to the senses. Later these ideas may be used as a foundation for an "apperceptive system." To produce clear knowledge in any science in its initial stages requires objective presentation. The extremes of no objective illustration and endless illustration should be avoided. A safe guide is the old rule to proceed from the known to the related unknown. When there is no known, objective illustration is essential. If there is clear knowledge from which to proceed, it is a waste of time and a loss of power not to use such knowledge. See Young, pp. 107, 209 to 212; Smith, pp. 101 to 109; McLellan and Dewey, p. 283; Bagley, Educative Process, pp. 254, 255; Suzzallo, pp. 42-59.

IV. ESSENTIALS FOR EFFICIENT RECALL.

To commit to memory and to be able to recall our knowledge when needed, certain mental laws must be observed. Among the essentials are strong impression, association, repetition, and frequent recall.

Few facts in arithmetic stand out so prominently in and of themselves as to make a strong and lasting impression at the first present-

ation. One fact is as important as another. Hence little use can be made of strong impressions. However, as the more concentrated the attention, the stronger the impression, the teacher must require the closest possible attention from every child. As the time of committing number facts is ordinarily during the age of passive or involuntary attention, and little reliance can be placed on forced attention the teacher must appeal to the child's instincts. She must appeal to his curiosity, satisfy his desire for new knowledge, for variety, for activity, for rhythm, for success. By means of games she may associate the work with child life. ✓

Drill, repetition, and frequent recall will be found necessary. All facts must be fully learned even at the cost of much drudgery. "It is safe to say that the point will never be reached where pain and drudgery can be entirely eliminated from the educative process", says Bagley. "If the pupil does not sometimes find his school work disagreeable, then something is radically wrong either with the pupil or with the school or with both." See Young, pp. 91 to 96. 135, 136, 251; Munsterberg, Psychology and the Teacher, Chap. XVI, and p. 122; De Garmo, Essentials of Method, p. 60; James, Psychology, Chap. XVIII; Radestock, Habit, pp. 32, 74; White, School Management, pp. 160-166; Bagley, op. cit., Chap. XI; Smith, Systematic Methodology, pp. 37, 38; Chamberlain, The Child, pp. 341, 342.

V. TEACHING A NEW NUMBER FACT—DRILLS.

In teaching a new number fact, as many different cortical areas as possible should be approached simultaneously or in quick succession, in order to make the proper nerve fiber connections in the brain as a basis for future association of ideas and quick recall. Approach visual, auditory, vocal muscular, and graphic muscular (strain) areas.

1. Teacher repeat.
2. Children repeat.
3. Teacher write on board.
4. Children read orally.
5. Teacher erase and children visualize.
6. Children repeat imaged fact.
7. Children copy written fact.
8. Children write imaged fact.

Repeat these processes to "form the habit" as it is commonly expressed, but speaking with physical accuracy, to form the proper nerve fiber connections. Have class repetition interspersed with individual repetition. Drill individuals found slow, then the class again. Give very slow pupils outside drill. No other work of any kind should be taken up after presenting new facts till these have been thorougly drilled upon. As already said all facts must be fully

learned even at the cost of much drudgery. Give a minute's absolute rest, then drill again.

The teacher need not be surprised if the last facts presented are forgotten before the next recitation. She must repeat the process and drill again. As often as it is found that some fact has been forgotten, drill must again be given on this particular fact. Before presenting new facts, be sure that the class know thoroughly all previously presented facts. Find what facts a child does not know and drill him accordingly. Responses should be practically instantaneous. "A child must not be allowed to forget old facts," says Munsterberg, "in the effort to acquire new facts." See Thompson, Brain and Personality; Munsterberg,op. cit., pp. 117, 118, 140-147; Suzzallo, Chap. VIII; Bailey, Teaching Arithmetic, p. 47; Bagley, pp. 122, 123, 328-331; McMurry, pp. 46, 56; James, Psychology, pp. 134-150, 298; Myers, Experimental Psychology, pp. 72-90, 112-116; Brown and Coffman, op. cit., Chap. VIII.

VI. REVIEWS.

We have noted that one of the essentials to memory is frequent recall. This is one, but only one, of the aims of reviews. There are **six purposes of reviews.**

1. That the teacher may learn the child's needs.
2. To show the child his needs.
3. To fix more firmly in mind (by frequent recall).
4. To organize materials—the general review.
5. As a preparation for new related knowledge—the first step of the Inductive Development Lesson.
6. To make knowledge usable. This, says Gordy, is the great function of reviews. (Lessons in Psychology.)

According to Gordy there are three stages of knowing:

1. Implicit knowledge—The child knows but cannot express what he knows.
2. Explicit knowledge—The child can tell what he knows, but he cannot use his knowledge.
3. Usable knowledge.—The child can use his knowledge as well as tell what he knows.

To show the importance of reviews, we may make use of the illustration of the snow storm.

On a winter morning we look out upon the streets to see that a great depth of snow has fallen. We watch the first team plod laboriously through to open a way. Another team follows, also with great difficulty, but each succeeding team finds the task easier, till at length the sleighs glide past with the greatest ease. But another storm fills up the broken way. The process of breaking out must be

repeated, but the difficulty is not so great as it was the first time and soon the sleighing is better than before. So storm after storm requires constant travel to keep the way open. If after the way has been once broken out, several storms occur without the street's being used in the meantime, the path may become utterly obliterated and forgotten.

So it is with the brain. Children, like adults, will forget when facts are allowed to remain unused for some time. The way must be broken out and rebroken; or to speak literally, the nerve fiber connections must be strengthened by repeated use.

Drills, therefore, should be given not only in connection with new work, but also in reviews. Drills should be quick and snappy. In review drills when the child hesitates, call on the class. Work rapidly while you work and waste no time. Ten minutes of rapid fire recitation is worth more than a half hour of dawdling.

See Munsterberg, op. cit., p. 122 and Chap. XVI; Smith, pp. 143, 144; Young, pp. 129, 142, 225, 226; McMurry, pp. 56, 80, 81, 98, 127, 136, 137, 140; McMurry, Method of the Recitation, pp. 114, 115; Radestock, Habit, p. 21; Bagley, op. cit. pp. 328-333.

VII. CONCERT RECITATIONS.

Beware of concert recitations. A skillful teacher may at times use concert recitations successfully.

1. In drill work. Here what is desired is repetition with attention, hence time is saved by having several repeat simultaneously.

2. When the answer is short and one child fails, the rest of the class may have the answer at their tongue's end.

3. When the answer is short and a pause is made after the question to allow all to grasp the meaning and form the correct answer. In recitation on number facts, for example, the teacher may point to successive facts on the board, pause briefly, say "class" or tap on the board with the pointer, and the class as a result of the pause is ready to answer simultaneously. If, however, no time is given for thought, one bright child will answer and the rest then repeat like parrots, some not even looking at the board to see what the fact is.

Where there is a possibility of different wordings to the answer, concert recitation is out of the question. See Page, Theory and Practice of Teaching, pp. 151, 152.

VIII. ATTENTION.

Keep every child attentive as long as he is supposed to be attentive. If impossible to hold the attention of the more advanced pupils while drilling backward pupils, set them at other work. Do not try to keep young children at high tension too long at a time. Allow them to rest, then begin again.

TO HOLD ATTENTION.

1. Secure interest if possible. In some way connect the work with child life. Games are often a great aid. But be sure to hold attention, interest or no interest. One of the great aims of education is the development of the power of voluntary, or active, attention. "May it not be", asks Munsterberg, "that the most important aim of education is just the power of overcoming the temptations of mere personal interest?" "An education", he says, "which simply follows the likings and interests, leaves the adolescent personality in a flabby and ineffective state." "There is no doubt that through the tendency of our times to yield to this demand for interesting instruction, we already feel the dangerous results of the crippling of the voluntary attention." "Public life has to suffer for it." "But the great word which is to control it is not pleasure but duty." Op. cit., pp. 16, 18, 190, 265, 269.

On the other hand, George Eliot says: "For getting a fine flourishing growth of stupidity there is nothing like pouring out on a mind a good amount of subjects in which it feels no interest. Mill on the Floss, p. 327.

2. Watch the class, not the pupil reciting.
3. Call on inattentive pupil when pupil reciting hesitates.
4. Call on the whole class when the pupil reciting hesitates, provided the answer is brief. If but few respond, it may be a sign of inattention. See De Garmo, Chap. XV; Munsterberg, op. cit., Chap. XVIII; James, Psychology, Chap. XIII; Radestock, Habit, pp. 70, 71; Puffer, Psychology of Beauty, pp. 160-168; Bagley, Classroom Management, pp. 137-187.

IX. THE ART OF QUESTIONING.

No teacher of Arithmetic should fail to make a thorough study of the art of questioning. The mere instructor has little need of questioning, but the true teacher must know how to question and how to question well. "To question well is to teach well," says DeGarmo. Those desiring to make themselves more proficient in this art are referred to the chapter on "The Art of Questioning" in DeGarmo's "Interest and Education". Also see Young, pp. 55, 67.

X. CLASS V. S. INDIVIDUAL RECITATION.

The teacher of power, like the conductor of the trained choir, carries along his whole class. Here may be a solo and there a duet followed by the ensemble, all being parts of one great production. So the teacher may call on this individual or on that to recite for the class and as a constituent part of the class, but the whole class is carried along as if it were a single individual. Every one is attentive and ready to respond if called upon. The individual is by no means neglected. The teacher watches every move, every expression even.

The slightest sign of incomprehension in a child's eye calls forth an enlightening illustration or a thought-provoking question.

However this complete class unity is found neither in the first recitation nor in the tenth. More or less individual drill must precede and accompany the choir practice. The quick ear of the conductor catches every weak point, and if he cannot strengthen it in general practice, he does it in private. So the successful class room teacher must supplement his class room work with individual instruction. This, in fact, is the strength of the Batavia System, in which one teacher does the class instruction and another the individual instruction.

But one might visit two adjoining rooms and hear the same question put and practically the same answer given by individual pupils; yet in one room the recitation might be class recitation and in the other individual recitation. That is, in the one room the recitation might be conducted in the manner described, while in the other only the person reciting may be paying attention. The class as a whole is receiving no benefit and the recitation consists of a series of individual recitations, logically connected in the teacher's mind perhaps, but disconnected and unrelated in the pupils' minds. See Young, 56, 57, 81 to 86, 138, 139; Bagley, Classroom Management Chap. XIV; Young, The Teaching of Mathematics in Prussia, pp. 56, 57.

XI. THE INDUCTIVE DEVELOPMENT LESSON.

Inductive reasoning consists in passing from particular truths to general truths. It is known as generalization. Induction is often the method used in presenting new matter to children. The child, in fact, begins his acquisition of knowledge by means of induction.

The inductive development lesson is generally considered as having five steps:

1. Preparation.
    Substep—Statement of the aim.
2. Presentation.
3. Comparison and Abstraction.
4. Generalization.
5. Application.

The preparation consists of a brief review of all matter necessary as a basis for the new work to be presented, in order that the child may have this knowledge fresh in his mind and thus readily take the step from the known to the related unknown. Not the whole subject should be reviewed, but only that necessary for the foundation. Generally a few questions will bring out all that is required. See McMurry, pp. 76, 79, 80; DeGarmo, Essentials of Method, pp. 46-51.

As to the statement of the aim, De Garmo says: "It would be unpedagogical not to have the pupil understand from the beginning what the aim of the lesson is." "A skillful dramatist never fully

reveals his plot ahead of its unfolding, nor does he, on the other hand ever allow any great but entirely unexpected culmination to occur." Essentials of Method, pp. 48, 50. "It should state as clearly as possible the point that the lesson is intended to make," says Bagley. "It should seize upon some need and show it may be satisfied." "The aim really forms the connecting link between the old and the new." Op. cit., pp. 291, 292. See McMurry 80, 81. However, many teachers of successful experience deliberately omit this sub-step.

The **presentation** consists in placing before the class several illustrations of the new matter to be taught, reaching it naturally from the matter reviewed in the preparation. The new matter may be brought out by questions, but occasionally it may be found best to show the connection directly without questions. See De Garmo, op. cit., pp. 51, 52.

The **comparison** and **abstraction** consists in noting what has been done in all examples.

The **generalization** consists in stating in the form of a general rule or formula how all similar examples should be worked. See McMurry, p. 69.

The **application** consists in working examples by the rule that has been generalized. This step as will be seen is deductive, though a part of the inductive development lesson. See Smith, pp. 111, 112; Bagley, op. cit., Chap. XIX. For lesson plans, see Stamper, A text book on the teaching of Arithmetic, Chap. VIII.

XII. TEACHING A NEW OPERATION.

In teaching a new operation, when possible, use the inductive process as above. But aside from this, when the class is large, there is a general method of procedure that will often be found advantageous.

1. In developing the process the teacher should work several easy examples on the board.
2. Send individuals to the board, the class observing and making suggestions under the direction of the teacher.
3. Send several pupils or the whole class to the board.
4. If several have trouble, send all to their seats and attack the difficulty again.
5. Send the class to the board again to finish the example.

Or in place of 4 and 5, as for example in teaching long division, the teacher may ask the class to work as she directs. Suppose each child has on the board ready for work the example 384 to be divided by 12. Teacher—How many times is 12 contained in 38? Where shall we place the 3? What shall we do with the 3 now? Where shall we place the 36? Etc. Several examples may be worked in this way until the class know all the steps.

6. Teacher give individual help to the few who still have trouble.

7. Slow pupils should be dealt with out of class to prevent the monopolizing of time that should be spent in class instruction.

It is important that **but one difficulty**, that is one new point, should be introduced at a time. "The importance of cutting work up into simple steps and taking them **one at a time**", says Young, "cannot be overestimated." (p. 128.) See Young, pp. 134, 135.

XIII. THE DEDUCTIVE DEVELOPMENT LESSON.

Deductive reasoning consists in passing from a general truth to a particular truth. We begin with rules and proceed to individual cases. Our first duty is to see that the **major premise** is true. In arithmetic we must apply the proper rule.

In Logic we have the Major Premise, the Minor Premise, and the Conclusion.

Major Premise: All A is B. Minor Premise: C is A. Conclusion: C is B.

The **Deductive Development Lesson** is considered as made up of four steps:

1. Statement of data. 2. Statement of governing principles or general rule. 3. Statement of the conclusion. 4. Verification.

It will be noted that the order of the logical sylogism and of the deductive development lesson as sometimes given is here reversed, the particular statement being given before the general. In promiscuous problem solving in arithmetic this inverted order is necessary. as the data is first stated and the pupil must determine what general rule applies to the particular case. He then draws his conclusion, states the problem accordingly, and solves. The Verification may be some check or reverse process, but it frequently consists in looking at the answer or in receiving the instructor's approval.

Example: A merchant invests \$160 in flour and sells it at a profit of \$40. Find the Rate of gain.

1. Statement of data: \$160 is Cost or B. \$40 is gain or P. Wanted—the Rate of gain or R.

2. Statement of general rule: P divided by B = R.

3. Statement of conclusion: \$40 ÷ \$160 = 25%. Rate of gain. Answer.

4. Verification: 25% of \$160 = \$40.

See Bagley, Educative Process, Chap. XX.

XIV. PROBLEM SOLVING.

There are several **difficulties** that the child will have to meet and remove before he can solve problems readily.

1. He must be able to identify the data and what is wanted.
2. He must determine what rule or formula to use.

3. He must determine what intermediate steps, if any, must be taken. Under this head he must determine:

(1) Is there a missing fact and what is it?

(2) Can the missing fact be found directly from the given data?

(3) If the fact cannot be thus found, what can be found from the given data, from which new data the missing fact may be found? This is an appeal to the puzzle instinct.

Much practice in identification of data in terms of Percentage, Mensuration, etc., should be given, independent of solution. At first it will be necessary to lead to identification by questioning. Then the child should be led to ask himself the questions. For a time he should be required to write on paper or on the board the data both in terms of the problem and in terms of Percentage, if that is the topic under study.

Even greater than identification of data may be the difficulty of determining and finding the missing fact.

Problem: If 4 hats cost $12, what will 7 hats cost? Here the missing fact is the cost of one hat. Question as to what must first be known before we can find the cost of 7 hats. Can we find that missing fact? How?

The following problem of exactly the same nature as the above sometimes puzzles eighth grade pupils: If $\frac{7}{8}$ of a bushel of wheat cost $\frac{3}{4}$ of a dollar, what will $\frac{2}{3}$ of a bushel cost? Using the former problem as a preparation determine the rule: divide the cost by the number to find the cost of one, and multiply this by the number of which the cost is desired. In both problems solve by cancellation.

Problem: What will 17 gallons of milk cost at 6 cents per quart? Here the essential missing fact may be given as either the price per gallon or the number of quarts. For further illustration, see Chapter on Percentage.

XV. TEACHING HOW TO STUDY.

One of the greatest, if not the greatest end of intellectual education, is to teach the child how to study independently of a teacher. Sometimes it seems that altogether too much teaching is being done and the pupil is becoming entirely dependent upon, instead of independent of, his teacher.

"Dependence is not the preparation for independence", says McMurry. "Indeed, great skill on the part of a teacher in these respects almost precludes such skill on the part of pupils. If allowed prominence year after year, it so undermines self-reliance that one's helplessness when alone is greatly increased." "By overlooking the difference between studying with a leader and alone, therefore, the teacher overlooks initiative, and in consequence, she not

only fails to develop that power, but she may easily undermine it by accustoming her pupils to dependence upon her." (How to study and teaching how to study, p. 290.)

Too much has the appeal been made to the ear. The child can make little or nothing from his text book without the teacher's first clearing up all difficulties for him in the "assignment". This is a fine thing for the dull pupil, but the bright child is weakened. Many of us were taught by the other extreme. The text book was placed in our hands and we were compelled to dig out the assigned lesson as best we could. This was fine for the strong, but woe be unto the weak. If he could not work an example and was compelled to call on his teacher, the latter would merely work out the example on a slate and return it without a word of explanation. A medial course is the best to follow. At first give sufficient assistance by means of suggestion and illustration, gradually decreasing the amount till the child is largely thrown upon his own responsibility.

A place where the child should early gain independence is in the solution of problems. This should first be taught by means of the Study Recitation, given for a time once a week perhaps. On other days the difficulties may be cleared up in the assignment by means of questions and illustrations as already indicated. But the child should be taught as soon as possible to ask himself the necessary questions without the aid of the teacher. "Power of initiative is the key to proper study", says McMurry. Op. cit., p. 288. See Page, op. cit., p. 45.

XVI. THE STUDY RECITATION.

The study recitation is at first given after the method of the deductive development lesson. The problems should for a time be placed on the board, but at length the regular text book lesson should be taken.

1. The children read the problem silently.

2. Children write on paper a statement of data and what is to be found. If in Percentage, identify in terms of Percentage. In the early lessons the teacher will find it necessary to question as to what each item in the problem is, for a time even having the answer given orally. Then the questions will be merely asked by the teacher and answered mentally unless some pupil indicates that he cannot identify a particular item. The teacher may ask how many can identify this item or that, or who cannot identify it; or she may merely ask whether there is anyone who cannot identify all the items. The teacher's questioning should become less and less, the children asking themselves the questions and gradually becoming more and more self reliant.

The teacher finally does nothing but pass among the pupils, glancing at their work and putting a question here and there as she finds the necessity. In time the pupils will take this step mentally

and put nothing on their papers, the great danger being that they will do so before they are strong enough. But this must be the ultimate aim.

3. Children determine how to find the result, whether it can be found directly or whether some intermediate step is required. This is the step in which the reasoning must take place. As in the former step the teacher will have to direct thought by means of questions. How do you find the result? Can you find it directly? What is the essential missing fact? Can you find that? What must you find first? Etc., etc. As before, the child should gradually learn to do his own questioning, the teacher's suggestions becoming rare.

4. The children state the problem.

5. Solve and place result in the statement.

6. Verify by some check if possible.

The Study Recitation should be given during the regular recitation period. For a time twice or at least once a week should be given to such work. Though it may seem impossible to take so much time from the ordinary recitation, yet in the end it will be found that much time has been saved.

When the child is able undirected to read and solve miscellaneous problems in Arithmetic, he has made a long step in the direction of one of the chief aims of school education: ability to educate himself from books without the actual bodily presence of a teacher. He is becoming a student. "When the principles can be explicitily stated and intelligently applied, the essential aim of arithmetic has been reached." "The intellectual treasures of the past lie locked up in books.. Proper school training unlocks this storehouse by accustoming one to their intelligent use." (McMurry, The Method of the Recitation, pp. 6, 142, 143.)

See Bagley, op. cit., pp. 316-322; Bagley, Classroom Management, pp. 206-210.

XVII. THE ASSIGNMENT.

By the Assignment as used by writers on pedagogy is meant not merely the lesson assigned for home study but the method of assigning the lesson. Bagley says: "This is a preliminary clearing of the road before the seat work begins." "The acme of a skillful assignment is reached when the teacher reveals just enough of what is contained in the lesson to stimulate in the pupils the desire to ascertain the rest for themselves." "In general, the assignment will be much more explicit and detailed in the intermediate grades, where the pupil is just learning to use text books, than in the upper grades and the high school, where some familiarity with the text book method may be assumed. But in all cases the assignment, whether it be brief or full, is an important step which should never be omitted." (Op. cit., pp. 317, 318.) The purpose of the assignment, therefore, is to clear up all insurmountable difficulties.

For example, suppose the lesson to be assigned consists of problems and that the class has had little work of the kind. The assignment should consist of such questions as are used in the Study Recitation. The study recitation prepares for this. Only a sufficient number of questions should be asked to enable the pupils to solve the problems without aid during the study period. The ability of the pupils to solve the problems will show the teacher how successful was her assignment. Some pupils will understand, or think they understand, under the questioning of the teacher; but when they come to take up their lesson alone, they find they are unable to solve the problem. Such pupils will need to be requestioned. As in the study lesson, the number of questions will decrease as the pupils gain in strength. This work should go hand in hand with the work of the study recitation, and in fact it is an essential feature of the method of teaching children how to study.

One danger that the inexperienced teacher is apt to fall into is that in the questioning in the study recitation and in the assignment, she will call on the brighter pupils, those who have little need of such help. When the questions are answered orally, those called upon should be the pupils meeting with difficulty. This is very different from the method followed in the regular recitation lesson when the aim is to learn how well the lesson has been prepared. In such recitations, good, bad, and indifferent must be called upon indiscriminately. See Young, pp. 132, 133, 147; De Garmo, Essentials of Method, p. 47; Bagley, op. cit., pp. 317- 319; Bagley, Classroom Management, pp 192-206.

XVIII. DICTATION.

Have all members of the class take down dictation promptly, accurately, and neatly. Dictate slowly enough for all to keep together, constantly increasing the speed till the whole class can take dictation rapidly.

XIX. ORAL AND SILENT MENTAL ARITHMETIC.

A large part of the time given to arithmetic should be devoted to oral work and to silent mental work in which only the answer is written. New work should generally be presented orally and the first written problems in each new topic should be as simple as the oral problems. In written work as few figures as possible should be used, short processes and silent mental computation supplementing the written. As a child works at the board, have him "chalk and talk"; that is, state orally each step as he writes it.

See Smith, p. 118; Young, pp. 134, 135, 230; Suzzallo, pp. 75-78; McMurry, pp. 57, 59, 80, 82, 98, 107-109, 123, 124, 127, 128, 133; Bailey, A Handy Book on Teaching Arithmetic, p. 50; Walsh, Methods in Arithmetic, pp. 25, 27; Young, Teaching of Mathematics in Prussia, p. 57-64; Munsterberg, op. cit., 283-285; Smith, The teaching of Arithmetic, Chap. VII. See Stamper, op. cit., pp. 228-233.

PART TWO

Primary Arithmetic

I. FIRST LESSONS IN NUMBER.

The first lessons in number should consist in finding the child's knowledge of indefinite relative magnitude and exact number, and his ability to count. He should be drilled in the use and meaning of such relative terms as short, shorter, shortest; long, tall; large, few, etc.

To find the exact knowledge of number, objects may be held up and the child called upon to tell how many there are, or he may be asked to bring a certain number of objects. He does not know the number six till he can both select six objects from a larger number of objects and also state correctly how many there are when six objects are shown to him. The knowledge that the individual child may have upon entering the first grade will depend upon his age, his home surroundings and training, and his kindergarten training. First grade teachers report varying degrees of knowledge from three to more than ten. In order to have a starting point, the common knowledge of the class must be learned, then the teaching may begin at this point.

There is some difference of opinion as to whether we should teach the numbers as far as ten, before teaching the figures representing them; or whether we should teach both number and figure at the same time. The problem is largely determined for us, however, with respect to the first numbers at least, by the child's learning the numbers without knowing the corresponding figures. The more common practice with regard to the remaining numbers is to follow the principle laid down by Quintilian regarding the teaching of the Roman alphabet, that is, to teach both name and figure at the same time.

The question as to the desirability of abstract counting is also largely predetermined. In school practice some of our best authorities hold that counting, to be of value, should be concrete and should follow the study of definite number so that when a child can count to twenty-five, he also knows twenty-five objects. It does not seem to be a matter of great moment; and whichever practice a teacher may follow, she has good authority to support her.

Taking for granted that we are to follow the practice of teaching all new numbers and the figures representing them at the same time, we shall find that our first lesson in exact number must be teaching the figures representing the numbers already known. This may be done by adapting the word method of teaching reading. It will doubtless be a saving of time to co-ordinate by teaching the writ-

ten word and the figure at the same time. In teaching the figure 3, supposing the class to be fully acquainted with the number three, there might be some question as to the necessity of presenting three objects. The question is whether the word "three" brings up a sufficiently vivid concept of the number three. The same question is presented in the teaching of the word "cat". Is it essential to present a picture of a cat, or does the word bring up a sufficiently vivid mental picture to do away with the need of the physical picture? Opinions of course will differ, but the question is worth considering. The general practice of those who believe in objective teaching would indicate a belief in the necessity of the physical picture.

In teaching a figure for a known number we may hold up the number of objects, elicit the name (or ask the children to bring three objects), write the figure on the board, have children name the figure and write it and bring the number of objects indicated. Drill.

In teaching a new number, as eight, supposing that some members of the class do not know the numbers beyond seven, teacher hold up eight objects and ask how many there are? Bring out from those who do not know the name that there is one more than seven. Have some child who knows tell how many this is. Write the figure 8 on the board and drill as before, giving special attention to those who did not know the number, and having them bring 8 objects, etc.

In teaching zero, teacher hold up her empty hand and ask how many objects she has in it. The class respond, "not any". Teacher write "0" on the board and tell class that this is how we express "not any" and that we call it naught or zero. Drill. Warning: Do not say nor allow children to say "ought".

See McLellan and Dewey, pp. 144-195; Brown and Coffman, 131-150; Smith, pp. 112-117; Walsh, pp. 33-45.

II. NUMBERS ABOVE NINE.

In teaching "ten" use the fingers on the two hands. Also use splints, tying ten in a bundle. Have several children stand and one child tell how many "ten fingers" there are. Teacher write 10 on the board and have children write it.

Teacher holding up one bundle of ten and one single splint, How many splints have I? As in eleven we have one ten and one single one, we write it thus, the left hand "one" tells us we have one "ten", and the right hand "one" tells us we have a single "one", or eleven in all. Similarly teach 12 and 13. If one ten and one is written thus, one ten and two thus, and one ten and three thus; who can tell how we shall write one ten and four? One ten and five? Etc.

Tell class that -teen means ten. If four and ten is called fourteen; six and ten, sixteen; seven and ten, seventeen; what shall we call eight and ten? Nine and ten?

Teacher holding up two bundles of ten, how many tens have I?

Obtain or give name. Etc. Similarly with twenty-one, twenty-two, and twenty-three. If twenty and one is called twenty-one; twenty and two, twenty-two; twenty and three, twenty-three; what shall we call twenty and four? Twenty and five? Etc. If twenty-one is written thus; twenty-two, thus; and twenty-three, thus; how shall we write twenty-four? Twenty-five? Etc. If one ten is written thus; two tens, thus; and three tens, thus; how shall we write four tens? Etc. If four tens is called forty; five tens, fifty; and six tens, sixty; what shall we call seven tens? Etc.

If the method of induction thus briefly sketched is used, the class should soon learn to name and write new numbers without the teacher's aid except by means of questioning. As the numbers are given by the child, the teacher should build up the following table. By having the known numbers thus systematically before the child, he will more readily determine how to write the new numbers.

1	11	21	31	41	51	61	71	81	91
2	12	22	32	42	52	62	72	82	92
3	13	23	33	43	53	63	73	83	93
4	14	24	34	44	54	64	74	84	94
5	15	25	35	45	55	65	75	85	95
6	16	26	36	46	56	66	76	86	96
7	17	27	37	47	57	67	77	87	97
8	18	28	38	48	58	68	78	88	98
9	19	29	39	49	59	69	79	89	99
10	20	30	40	50	60	70	80	90	100

It is important that only such numbers as the class know be placed before them in the above table. Similarly teach the numbers to 1000.' See Speer's Elementary Arithmetic; p. 125.

III. READING AND WRITING NUMBERS ABOVE 1000.

1. Instead of building up the numbers in regular order above 1000, begin by teaching the reading of such numbers as 371,371 and follow this by the writing of such numbers.

2. Next teach reading followed by writing such numbers as 235,476; 46,840; 2,896. Emphasize that the three right hand digits represent units and the next three digits to the left represent thousands. Place the word thousand above the second period and units above the first period and have the number read for a time as follows: 235 thousand, 476 units, soon dropping the word 'units''.

3. Teach such numbers as 234,054; 5,023; 87,004; at first reading and dictating as follows: 5 thousand, no hundred 23; 87 thousand, no hundred no tens 4; thus emphasizing the ciphers and the fact that there must be three places to the right of each comma. Frequent drill in writing and reading such numbers will be necessary for several terms after the work is first taught.

4. Teach millions as above.

After a little practice of this sort, dictate in the usual way, at first questioning as to what should be placed in hundreds place, etc.

IV. ADDITION.

In arriving at and learning the addition facts the child should under no circumstances be allowed to count, therefore do not use single objects in teaching these facts. Some teachers will cry out against this as a great heresy in that it violates the principle of objective teaching. This, however, is an error, but even though it were true, the evils resulting from the objective method in this particular instance far outweigh its benefits if there be any benefits. The child who learns his facts by counting, generally never learns his facts, paradoxical as that may sound. When he has a column of figures to add, he continues to count through life. This is a most pernicious habit and results in slow and inaccurate computation.

However, there is a means of discovering a fact without developing the pernicious counting habit. It is an adaptation of the method used by Montessori. Have wooden rods of 1 inch, 2 inches, etc., up to 18 inches in length with the length written upon them in figures. **The rods should not be marked off in inch lengths** as this would, as in the case of single objects, lead to counting. To teach the sum of two numbers, as 2 plus 3, have each child find a two inch and a three inch rod and place them end to end and then find another rod equal in length to the sum of the two rods.

2. Having thus discovered that 2 plus 3 equals 5, approach all the cortical areas of the brain as indicated in the discussion regarding the teaching of new number facts. Teacher repeat, children repeat, teacher write on the board, children visualize, children copy. Many children have to repeat the facts before they can give the results. They should be drilled to give answers at sight or sound as well as by repetition.

3. Teach both $2+3$ and $3+2$ at the same time, as this is but one fact.

4. Teach both horizontally and in columns.

5. Do not try to teach any new facts till all the old facts have been thoroughly learned.

6. The order of teaching facts should be determined by sums rather than by tables. Teach two plus two, then all the facts whose sum is five, then those whose sum is six, etc.

7. Preparation for subtraction should go hand in hand with the learning of the addition facts. See discussion of subtraction.

8. After a few facts have been learned, column addition with sum not greater than nine should be begun. Also simple oral examples such as 2 plus 2 plus 2. During the second half year column addition should be continued to sum 18, but in no case should the column exceed the number facts already learned. For example in the

column 3-6-6, if the class know the facts only so far as sum 15, they could add downward but not upward; as in one case the greatest fact is 9 plus 6 which they have learned, while in the other case the fact is 12 plus 3 which is not even one of the 49 elementary number facts, which include only the facts from 1 plus 1 to 9 plus 9. In column addition do not point or allow children to point. This is important.

9. Carrying may alse be taught during the second term, the New York State Syllabus advising no explanation. Always add the carry **immediately**. Do not write it at the top of the next column, but add it to the top digit at once if adding down, and to the bottom digit if adding up.

10. Each day the previously learned facts should be reviewed by various means: by placing the facts on the board, the child to fill in the results orally or on the board, or by copying on paper; by placing a number in a hollow square or within a circle of numbers and adding it in turn to each of the outer numbers without any pointing; by using number cards; by means of ladders, steps, guessing games, and other devices that may occur to the teacher.

As many of the class as possible should try to reach the top of ladder or steps without missing a step. The number cards should have on one side the reverse of the combination found on the other side; that is, if 4 over 8 is on one side, 8 over 4 should be on the other side. Thus the teacher can see the fact on the back of the card as she brings it from the back of the pack to the front.. This is important. She should stand where all the class can see the cards without effort and should manipulate them rapidly without fumbling. She should vary the method of recitation; sometimes calling on the children by name for successive facts, rarely calling on them in order, generally having one child rise and recite several facts before calling on another pupil. When one child misses, say "class'" or "tell', and the child missing should pass to the board and write the fact in the four ways for writing each new fact. Children should recite these facts rapidly. DO NOT GIVE THEM TIME TO OBTAIN THE RESULT BY COUNTING. Another device is to place the missed fact back in the pack where it will occur again soon and thus give the child another chance to recognize the fact. Another device is to hand the card to the child to study. Very little time, perhaps from three to five minutes a day, should be spent in this drill.

In the guessing game the teacher or one of the pupils says: "I am thinking of two numbers whose sum is 14." The class then guess as called upon by the teacher, "Is it 7 and 7 are 14?" "Is it 8 and 6 are 14?" etc., till the correct combination is guessed. It is generally well for the child to tell the teacher what fact he has chosen. As games take time, they should be used with discretion.

After carrying has been taught, see that all examples given before beginning series illustrate both carrying and not carrying, that

the children may constantly be compelled to judge whether they must carry or not.

25224	74476	34332	63648	34716
64454	21320	24023	34231	53243
23234	43553	85579	71587	98929

Have class rise, add down, and be seated as fast as the result of a column is found. Calling on some child, often the one seated last, ask the result. The child may answer in the first example, second column, "ten". As if the teacher considered this result correct, she should ask how many have ten. Why not, Charles? Because there is one to carry. Add it again orally, Frank. See that the carry is made at once whether beginning with the bottom or top figure. This cannot be over emphasized. In the fourth column, Frank may this time give the result as thirteen. Question as before, bringing out that there is no carry this time. This method will keep the children constantly on the alert to see whether there is a carry. After a few minutes of this practice, send the class to the board to work examples already on the board. As fast as two children have their results correct, have them erase results and exchange examples, thus keeping all busy.

One point of importance that all teachers should observe is to have chalk and erasers ready at the board before class that no time need be wasted in class.

See Brown and Coffman, op. cit., pp. 150-159; Bailey, op. cit., Lesson 16; Walsh, Methods in Arithmetic, Chap. II; McLellan and Dewey, pp. 195-200; Smith, pp. 114-117; McMurry, pp. 30-53.

V. SERIES AND COLUMN ADDITION.

There are three classes of series:

1. Sum of right hand digits less than ten. Illustration: sum of right hand digits 7.

24	44	84	64	73	33	53	94	14	
3	3	3	3	4	4	4	3	3	etc.

2. Sum of right hand digits ten.

28 plus 2, 38 plus 2, 42 plus 8, etc.

3. Sum of right hand digits more than ten. Illustration, sum of right hand figures 12:

27 plus 5, 77 plus 5, 85 plus 7, etc.

The teacher may give the series orally, the children giving the results; the series may be placed vertically on the board as above; or the single digit number may be placed in the center of a series of two digit numbers, all having the same right hand digit, the child giving results in regular order from right to left or left to right, no pointing. In the oral work the teacher should say 27-5. If the child hesitates, the teacher should say 7-5, and the child should say 32. If he says 12,

the teacher should repeat 27-5. If the child still hesitates, the teacher should return to 7-5, and on receiving the reply 12 again, the teacher should say, what is the right hand figure? Then what is the right hand figure in the sum of 27 and 5? 27-5? If the child still cannot give the result place the numbers vertically on the board and add as in column addition.

One purpose of series drill is preparation for column addition. The two should therefore go hand in hand. Children who may know readily that 7 plus 5 is 12, may not know without counting that 27 plus 5 is 32. In the column example 6-8-3 the child may start correctly by saying 14 (he should never say 6 plus 8 are 14, or even 6-14, but only 14) and then, though he may know that 4 and 3 are 7, he may not know that 14 and 3 are 17. The teacher should say "4-3". If the child says "17", do not compel him to say "7", as 17 was the answer wanted. If he says "7", you may be compelled to say, "Then how much is 14 and 3"? If the series 4-3 is on the board at the time, the teacher may merely refer to this.

During the third half year, series, sum less than 10 should begin. Such work should at first be oral, followed with blackboard work similar to the illustrations given above, and accompanied by appropriate column addition. The work should continue as necessary through the following grades.

The series should be built up something as follows: 0 plus 1 to 0 plus 9; 1 plus 1; 1 plus 2; etc. to 1 plus 9; miscellaneous practice. The child's readiness in the miscellaneous practice and column addition will show how successful the series has been. Follow this with 2 plus 2; miscellaneous practice applying all series hitherto drilled on; 2 plus 3; miscellaneous; 3 plus 3; miscellaneous, etc., up to 4 plus 5.

As soon as series 2 plus 2 has been learned, such examples as the following should be given. The children should add from the top down as follows: 12-14-15; 5-7-12-14; 9-12-13-14. The addition should be oral that the teacher my know how the pupil is arriving at the result. No other numbers or words than those used above should be used by the children in adding. If the examples are added from the bottom up, they will not be in accord with the series.

8	4	6
3	2	6
1	5	2
1	2	1

FIRST SERIES—SUM OF RIGHT HAND DIGITS LESS THAN TEN.

The following examples are constructed progressively according to the series:

347	245	274	789	456	264	639	357
653	755	726	211	544	736	361	643
876	318	539	423	543	589	986	518

243	148	431	142	246	348	768	648
867	962	679	899	854	853	343	462
649	871	572	516	765	175	678	631
110	16	110	111	111	110	110	118

579	594	908	870
643	426	294	948
222	222	272	124
101	212	111	12

876	543	375	294	206	458	902	980
456	777	777	938	885	884	399	339
223	233	213	322	141	213	231	222
101	112	211	110	323	111	123	117

987	876	567	452	378	597	397	958
357	555	777	777	666	645	824	463
122	211	122	413	222	424	441	241
211	124	211	124	211	100	114	114

767	898	858	375
574	225	172	848
322	243	333	413
113	311	313	131

397	894	836	975	789	597	345	738
823	629	696	547	743	655	667	589
532	232	225	235	235	423	245	331
125	122	121	121	111	112	521	121

489	638	543	178	837	957	839	847
954	696	779	977	586	585	694	598
334	323	225	322	224	235	214	322
110	101	231	301	121	101	131	110

876	593	956	147
567	749	568	996
434	242	224	341
110	214	141	414

849	849	748	847	958	678	567	596
773	273	498	389	264	655	655	466
264	146	422	442	656	323	604	224
112	621	221	110	101	222	162	612

847	876	638	938	538	678	587	596
683	349	677	294	746	746	765	847
353	313	223	153	571	441	531	245
115	351	351	513	133	123	113	211

847	948	657	847	673	537	586	509
465	484	746	674	668	777	636	894
447	237	324	301	531	513	473	454
230	220	272	177	127	172	203	132

976	849	173	438	838	489	473	821
678	573	987	974	786	856	792	599
312	216	433	225	214	342	428	243
33	361	306	362	161	312	306	326

678	678	483	459	543	678	594	731
954	542	534	497	537	853	450	947
121	434	438	432	464	434	429	286
245	245	543	501	454	24	523	34

SECOND SERIES—SUM OF RIGHT HAND DIGITS TEN.

At first the class should find any two successive digits whose sum is ten and the teacher or child should connect them with a curve. Then for a time the tens should be found before adding, but the curves should be dropped. As soon as possible the child should acquire the ability of finding the combinations of ten as he adds. Note that in these examples the passage from the teens to the twenties and from the twenties to the thirties, etc., is made by a combination of 10. Before beginning the column addition, give series drill as before on the combination to be introduced in the example.

The child should add rapidly as follows (see first two examples): 15-20-23-33-37; 9-15-20-26-36-38; 12-22-25-30-38; 11-15-20-24-34-39; 11-21-25-30-38. 13-15-20-29; 12-15-20-27-37; 11-21-25-30-38; 9-15-20-24-34-37; 11-15-20-28.

88968	46853	58947	78948	29498
54567	46555	85389	65698	14969
55555	4[illegible]535	65464	65445	76743
44363	54252	42444	45345	35373
55555	25275	44444	44454	37373
75455	25554	46646	32454	33577
15434	43855	35666	57365	43437

49935	94886	95498	98657	85899
96778	67386	78888	69894	89279
34773	63788	24282	15219	29612
37577	42878	28272	86119	91981
56567	24836	88738	84778	72727
57344	54724	82822	24612	95393
73934	36372	52297	21491	13179

THIRD SERIES—SUM OF RIGHT HAND DIGITS GREATER THAN TEN.

First give series and examples in which nine is added to various numbers, calling attention to the fact that the right hand digit of the result is one less than, and the left hand digit one greater than the corresponding digits of the number added to nine. Then give the series in the order illustrated in the following examples:

98788	48398	91398	99799	24898
79998	84638	28312	99789	68688
35989	56435	42777	12893	46656
99875	44574	67464	82286	73466
67998	47757	75669	78336	33347
93949	63763	96447	78844	64563
29489	66326	94744	37044	76663

42348	89299	78989	39769	99248
72689	75859	93948	88349	88278
78465	59874	76846	28885	47832
57558	71635	64765	55898	86866
79385	34646	87677	18684	83467
34627	89494	65183	88688	26545
57656	64475	64874	87858	88888

43438	99438
48768	45766
62342	88344
68687	98888
85455	86454
68787	58888
77878	88788

VI. SUBTRACTION.

Teach subtraction by the Austrian or addition process. The following preparation should be given while teaching the addition facts: 6 + ? = 9; 3 + ? = 9. Arrange on board or cards both horizontally and vertically.

Bring out the subtraction idea concretely. If you had 4 apples and gave me 2, how many would you have left? How many do you

have to add to 2 to have 4? Then 4—2=? If you have 5 marbles and lose 2, how many do you have left? How many do you add to 2 to have 5? Then 5—2=?

Give several examples like the above that have come within the actual experience of the child and bring out that we may find the result of subtraction by adding to the subtrahend a number that will give the minuend. Make constant use of the terms subtraction, minuend, subtrahend, and difference, that the children may become thoroughly familiar with their meaning. It is much better to be able to use the terms accurately without hesitation than to learn the definition without making use of the term itself in the work. Thus, in fact, the child has learned the content of most of the words he uses. Comparatively few words have been actually defined for him.

For practice give exercises as follows:

1. Each digit in the subtrahend equal to, or less than, the corresponding digit in the minuend.

$$\begin{array}{r} 4\,7\,9\,4\,8\,6\,3\,5\,8\,6\,7\,2\,0\,8 \\ -\ 1\,5\,2\,3\,0\,6\,1\,2\,3\,4\,2\,1\,0\,4 \end{array}$$

Emphasize that we add down in subtraction; that is, we add the lower number, or the subtrahend, to the number that we place below, or the difference, in order to give the minuend, or the upper number. At first the teacher will have to question the child as to how much must be added to 4 to give 8, etc. Then the child should ask himself these questions, but as soon as possible he should add as follows without any extra words: 4 plus 4 is 8; 0 plus 0 is 0; 1 plus 1 is 2; 2 plus 5 is 7; etc.; putting down result as he repeats it. Even this should be abandoned and the result seen at a glance and put down without a word. Not much time should be spent on this step.

2. Digits in the minuend less than the corresponding digits in the subtrahend. If the results of the first exercises are written before the subtraction begins, the method will be seen more clearly.

$$\begin{array}{r} 7\,3\,4\,6\,9\,5\,0\,3\,7\,8\,2\,6 \\ -\ 1\,7\,8\,9\,2\,3\,6\,1\,3\,9\,1\,7 \\ \hline 5\,5\,5\,7\,7\,1\,4\,2\,3\,9\,0\,9 \end{array}$$

As in the case of smaller numbers, we add the subtrahend and difference to obtain the minuend. Thus: 7 plus 9 is 16. In addition we should put down the 6 and carry the 1, but the 6 is already written in the minuend, so all we have to do after setting down the 9 is to carry the 1 exactly as we do in addition. Adding the 1 that we carry to the 1 in the subtrahend gives us 2 and we say, 2 plus 0 is 2. Just as in addition we add the number carried to the top addend without any extra words, merely giving the result of such addition, so here we merely add the amount to be carried to the subtrahend, which in this case is equivalent to the top addend. We then add this result to

the difference, which is the equivalent to the next addend. Thus, 9 plus 9 is 18; 4 plus 3 is 7; 1 plus 2 is 3; 6 plus 4 is 10; 4 plus 1 is 5; 2 plus 7 is 9; 9 plus 7 is 16; 9 plus 5 is 14; 8 plus 5 is 13: 2 plus 5 is 7.

Call attention to the fact that when the digit in the subtrahend is greater than that in the minuend, we add enough to give a number whose right hand digit is the digit in the minuend. That is, we cannot add to 7 to give 6 therefore we add enough to give 16. When we do this, we must carry the 1. After going over several examples that the teacher has solved on the board, place on the board exercises to be solved.

$$
\begin{array}{r}
5\,4\,8\,9\,2\,0\,7\,0\,3\,1\,8 \\
-\;1\,7\,9\,2\,5\,3\,7\,5\,2\,3\,6
\end{array}
$$

The first 2 will be given without difficulty. Teacher question as follows: Can we add to 3 to give 1? We must add enough to give how much? Ans. 11. How much must we add to 3 to give 11? Teacher or child sets down 8. As the figure in the minuend is only 1 instead of 11, what shall we do? Ans. Carry 1. To what shall we add it? Then 3 and how many make 3? How many to carry? Why not? What shall we say next? Ans. 5 plus 9 is 14. Is there anything to carry? Etc. After working several thus with the class, send individuals to the board to work under the direction of the teacher with suggestions by the rest of the class; then send the whole class to the board, using the "**chalk and talk**" method. Be sure that the voice does not indicate the answer to the question whether there is anything to carry.

From now on every example should illustrate both carrying and not carrying in irregular order, that the child may be compelled constantly to determine whether there is a carry or not.

As soon as possible the class should add rapidly as follows: 6 plus 2 is 8; 3 plus 8 is 11; 3 plus 0 is 3; 5 plus 9 is 14; 8 plus 9 is 17; 4 plus 6 is 10; etc. At length the result should be seen at a glance.

The advantage of the addition method is that only one set of facts has to be learned instead of two. If the teacher is not watchful, however, one drawback will be found: in the solution of problems requiring subtraction, the subtraction idea will be lost sight of. The teacher must emphasize the fact that subtraction means take away and that we merely find how much to take away by means of the addition facts. Simple problems in subtraction should be given from the first to obviate this difficulty.

See Brown and Coffman, pp. 160-163; Walsh, Chap. III; Bailey, Lesson 17; Smith, pp. 121-122; McLellan and Dewey, pp. 200-206.

VII. MULTIPLICATION.

No attempt to teach multiplication should be made until all the addition facts have been thoroughly learned. Teaching the four fun-

damental operations simultaneously is the great weakness as well as the chief feature of the Grube method.

McMurry (p. 54) suggests the following order of teaching the multiplication tables: "10s, 2s, 5s, 4s, 8s, 3s, 6s, 9s, 7s." This is an improvement over the old order.

In an excellent little monograph on teaching "The multiplication tables", Gildemeister uses the following order: 2s, 1s, 10s, 11s, 9s, 5s, 0s, 3s, 4s, 6s, 7s, 8s, 12s. A study of this pamphlet will repay any teacher of multiplication, though it proposes a too early introduction of division.

At first the facts should be found inductively. To teach $2\times 3=6$, give 3 plus 3 is 6, that is, two 3s are 6, or 2 times 3 is 6. 2 plus 2 plus 2 is 6; that is, three 2s are 6, or 3 times 2 is 6. Teach 2×3 and 3×2 as one fact, writing both horizontally and vertically. See p. 6 for method of teaching a new number fact.

Concentrate upon one fact till that is known. Sooner or later the child must learn each fact as an individual fact, or he will not learn it at all. As in addition, drill by means of oral questions, cards, facts on board or paper, circle, etc. Bring together such facts as 3×4 and 2×6. Give table of squares as 2×2, 3×3, etc. Preparation for division, as $3\times ?=6$, should go hand in hand with multiplication.

When assigned facts have been written out by the pupil, a chart of the multiplication tables may be placed before him that he may compare his results with it to see whether his are correct. He may thus make his own corrections and study upon those missed, or skipped because he was in doubt about them. The experience of many teachers has been that when the only means taken to teach the tables is by hanging the chart before the class while examples are to be worked, children generally depend entirely upon the chart and do not commit the facts. As indicated above, the facts should first be written and the chart merely used for correction.

As soon as a few facts have been learned, they should be used in examples. The teacher should at first place the examples on the board and write the results as the children give them. As soon as 2×3 is reached, the following example may be used: 2×213. Send class to board. Time may be saved by multiplying the result of one multiplication by another multiplier.

As carrying has already been taught in addition, the idea ought to present little difficulty here. As an aid to carrying, daily drill should be given with such exercises as $4\times 7+5$; $7\times 8+6$. This drill should continue till the most difficult operations in multiplication can be performed with facility.

See that the children use readily the terms **multiplication, multiplier, multiplicand,** and **product**. From the beginning give simple problems applying the principle of multiplication. At first oral prob-

lems should be given. When written problems of any type are introduced, the numbers used should be as small as those used in the oral problems. The child can thus give his entire attention to the new difficulty of expressing in writing what he has heretofore expressed orally, without being distracted by the size of the numbers which may be gradually increased after he has become familiar with the method. That the first written solution of a problem and the use of large numbers in solutions are both serious difficulties will be the testimony of any teacher of experience. Adhere rigidly to the pedagogical principle to introduce but one difficulty at a time. The violation of this rule inevitably leads to confusion. In the working of problems emphasize that the product is always the same in kind as the multiplicand. If we multiply apples, the result is apples; if we multiply marbles, the result is marbles; if we multiply an abstract number, the result is an abstract number. $2 \times \$3$ should be read 2 times \$3. $\$3 \times 2$ should be read \$3 multiplied by 2.

Teach early multiplication by 10 by merely annexing a cipher to the multiplicand. Give much practice in applying this short process and allow no other. Later give similar practice in multiplying by powers of 10. Never allow a child to multiply through by zero and place a row of zeroes as a partial product.

Teach multiplication by the factors of a number in varied order. $30 \times \$25 = 2 \times 3 \times 5 \times \$25 = 3 \times 5 \times 2 \times \$25 = 2 \times 5 \times 3 \times \$25$.

See Brown and Coffman, pp. 160-167; Bailey, Lesson 18; Walsh, Chap. IV; McLellan and Dewey, pp. 207-220.

VIII. DIVISION.

Division and Partition. Whether or not the terms division and partition are both employed, the distinction in ideas implied should be made. That is, in multiplication the product is the result of multiplying together two factors, one of which may be concrete and at least one abstract. When the process is reversed in division, the product becomes the dividend, one of the factors the divisor, and the other factor the quotient. If \$15 is the product of 3 and of \$5, then \$15 may be divided by \$5 or by 3. \$15 divided by \$5 equals 3, and \$15 divided by 3 equals \$5. Much practice should be given in determining whether the quotient is abstract or concrete.

After division is well in hand, it is well to check examples by multiplying the quotient by the divisor. Multiplication examples also should be checked by dividing the product by the multiplier, not by the multiplicand as the child may then merely copy the mistakes he may have made in his multiplication.

For checks of various kinds, see Brown and Coffman, Chap. IV.

Simple division problems should be used from the first. Also use and have children use terms division, dividend, divisor, quotient, remainder, and trial divisor.

Teach division as the reverse of multiplication. As a preparation while teaching the multiplication tables have the child for example tell by what we must multiply 7 to give 28. $7 \times ? = 28$. $4 \times ? = 28$. Then when we come to division, we refer to this and bring out that 28 divided by 4 gives 7. As in subtraction, so here we must guard against the danger of not grasping the division idea. To obviate this difficulty, employ many simple problems within the experience and grasp of the child.

In division as in addition it is of the greatest importance to take one step at a time.. The following series of progressive steps has worked itself out as the result of class room observation.

IX. SHORT DIVISION.

In both short and long division, place the quotient above the dividend. The first quotient figure should be directly above the right hand figure of the first partial dividend. Each succeeding figure in the dividend should have a corresponding figure above it in the quotient. **Insist on exact placing of quotient figures.**

First teach division within the multiplication tables. (a) Divisor exactly contained in the dividend; as, 7)14, 7)21, etc. (b) Division with a remainder; as, 7)16, 7)69, etc. Before giving more difficult examples, give thorough drill in dividing any number below 20 by 2, any number below 30 by 3, etc., up to 89 divided by 9. Write the examples on board or give them orally, having child give quotient and remainder orally, or send class to the board to work several examples as rapidly as possible. The teacher may place several examples at each child's place before class, or she may place them on some conspicuous board or chart, numbering them in regular order. Number the children in irregular order to prevent copying; as, 1, 4, 7, 10, 2, 5, 8, 11, 3, 6, 9, 12. Have each child begin with the example corresponding with his number, and as soon as he works one example, begin the next one without waiting. Those beginning with the higher numbers should turn back to example 1 as soon as they finish the last one given. The teacher should pass around giving a "C" for each correct, and a "0" for each incorrect result. An honor roll may be kept of those working a certain number without error. Another device is to place the divisor within a circle of dividends. This device should be continued for rapid drill even after formal work in short division begins.

After division within the tables is well in hand, give thorough drill with examples of progressive difficulty as follows: 4)4884, 3)30639, 7)14707, 6)121824, 8)168056, 4)487, 5)105157, 4)52804, 3)741. Give easy divisors at first, gradually increasing their difficulty. For an abundance of examples illustrating most of these type examples, see Woodfield, A manual on the teaching of division.

As a preparation for long division, teach short division by 11

and 12. Teach division by 10 by merely striking off the right hand figure, the left hand figures being the quotient and the right hand figure the remainder.

X. LONG DIVISION.

Never allow long division with divisors less than 11. Place the result above the dividend.

1. Division by 11 and 12. As a preparation use short division with divisors 11 and 12. Have a child tell how he worked a certain example and the teacher put down all the work that the child has done mentally, calling attention to the successive steps. After working several examples thus, send a child to the board to go through the written steps under the guidance of the teacher. Finally send the whole class to the board and use the "chalk and talk" method. Suppose the example to be 12)168. How many times is 12 contained in 16, Mary? Once. Where do we write the 1? Above the 6. Write it, class. 1 times 12, James? Where shall we write the 12? Write it, class. What shall we do next, Anna? Bring down the 8. Class brings down the 8. 12 is in 48, Harold? Etc. Continue this process till the class can take the steps unaided. Use examples that will give but two places in the result. Continue with divisors 11 and 12 till the class is familiar with the order of the steps: 1) divide, 2.) multiply, 3.) subtract, 4.) bring down, repeat.

2. Each quotient figure found by dividing the first digit or first two digits of the partial dividend by the first digit of the divisor, called the trial divisor. (See Shutts, Handbook of Arithmetic, p. 27). The following examples illustrate this step: Divisor 13, numbers from 143 to 149, 156 to 159, 260-269, 273-9, 286-9; divisor 14, 140-9, 154-9, 168-9, 294-9; divisor 21, 231-9, 441-9, 651-9, 861-9, 252-9, 462-9, 672-9, 882-9, 273-9, 483-9, 504-9, 714-9, 903-9, 924-9; 22), 242-9, 264-9, 286-9, 462-9, 484-9, 682-9; 23), 253-9, 276-9, 483-9, 506-9, 713-9, 943-9, 966-9; 24), 264-9, 288-9; 31), 341-9, 651-9, 961-9, 372-9, 682-9, 992-9, 401-9, 713-9, 961-9, 992-9, 32), 352-9, 672-9, 384-9, 416-9, 704-9, 736-9; 33), 363-9, 396-9, 693-9, 726-9; 34), 374-9, 680-9, 714-9; 35), 420-9, 735-9, 770-9; 36), 432-9, 756-9, 792-9; 37), 404-9, 444-9, 777-9, 814-9; 41), 451-9, 861-9, 492-9, 820-9; 42), 462-9, 882-9; 43), 473-9; 44), 484-9; 21), 1071-9, 1260-9, 1281-9, 1470-9, 1491-9 1113-9, 1134-9, 1302-9, 1323-9, 1554-9, 1743-9, 1932-9, 1974-9; 31), 1271-9, 1550-9, 1581-9, 1333-9, 1643-9, 1922-9, 2232-9, 2573-9, 2883-9; 41), 1271-9, 1681-9, 1722-9, 2132-9, 2296-9, 2337-9; 51), 1071-9, 1581-9; 52), 1092-9; 61), 1281-9, 1891-9; 71), 1491-9. For further examples and fuller development of this topic, see Woodfield, op. cit., pp. 12-22.

3. Quotient figure one less than the number of times the trial divisor is contained in the trial dividend. Here emphasize the two principles: (a) The subtrahend should not be greater than the partial dividend. If it is, it is a sign that the quotient figure should be

smaller. (b) The remainder should not be greater than, or equal to, the divisor. If it is, it is a sign that the quotient figure should be larger. Examples: divisor 22), 400-417, 600-615, 800-813; 23), 400-413, 600-619, 800-804, 810-827; 24), 400-419, 600-619, 800-815; 25), 400-409, 600-619, 810-824; 26), 416-419, 600-619, 820-831; 32), 1230-1247, 1504-1535, 1810-1823; 31), 1510-8, 1820-8; 33), 1200-1220, 1230-1253, 1500-1517; 43), 1610-1633, 1650-1676; 44), 2120-2155, 2920-2947; 46), 3960-3999; 47), 2980-3007; 54), 3000-3023.

4. Quotient figure more than one less than the number of times the trial divisor is contained in the trial dividend. Here the divisors from 14 to 19 should be used. As 19 is nearer 20 in value than it is to 10, take 2 instead of 1 as the trial divisor. With divisors 14 to 18, take both 1 and 2 as trial divisors and try as quotient figure some digit between the two results of trial. Here it must be noted that in dividing such numbers as 1288 by 14, 1 is not the trial dividend as 12 is less than 14. In such cases, as 9 is the largest digit, we try 9 or some smaller number as the trial quotient.

5. More than two figures in the quotient. Promiscuous examples with two figures in the divisor may now be taken up. These may be found in any text book or taken from such cards as Maxson's self-keyed number cards.

6. More than two figures in the divisor. In dividing by such numbers as 20, 300, 4000, etc., do not allow long division. Strike off as a remainder as many places in the dividend as there are ciphers at the right of the significant figure of the divisor, and divide by short division. If there is an extra remainder after dividing by the digit, place this before the remainder struck off.

124	178	8
6/)744/0	4/00)712/45	8/000)65/721—1721

See Brown and Coffman, pp. 167-170; Bailey, Lessons 19, 20; McLellan and Dewey, pp. 119-143, 220-240; McMurray, pp. 60-79; Walsh, pp. 90-105.

X. ROMAN NOTATION.

Systematize the method of teaching the Roman notation. First teach the following:

1=I, 10=X, 100=C, 1000=M.

After this is thoroughly learned, give the following:

2=II.	20=XX,	200=CC,	2000=MM.
3=III,	30=XXX,	300=CCC,	3000=MMM.

Next give:

5=V, 50=L, 500=D, 5000=$\overline{V}$

Follow with:

6=VI,	60=LX,	600=DC,	6000=$\overline{VI}$
7=VII,	70=LXX,	700=DCC,	7000=$\overline{VII}$
8=VIII,	80=LXXX,	800=DCCC,	8000=$\overline{VIII}$

This completes the method by addition. Now give the method by subtraction:

4=IV,	40=XL,	400=CD,	4000=$\overline{\text{IV}}$
9=IX,	90=XC,	900=CM,	9000=$\overline{\text{IX}}$

The above thirty-six numbers include every possible combination below 10,000. When the child can write these numbers readily, he should have no difficulty with combinations of two or more of them.

Example. Write 3469 in Roman notation. 3469=3000+400+60+9.·.3469=MMMCDLXIX. 1912=MCMXII.

Ones and fives have characters of their own. Twos, threes, sixes, sevens, and eights are written by addition. Fours and nines are written by subtraction.

Rule: **Beginning at the left, write in Roman characters the number represented by each digit in turn. That is, while writing the equivalent to one digit, do not pay any attention to the other digits.**

XI. CANCELLATION.

Teach cancellation progressively as follows. See illustrations below. 1. The same numbers in dividend and divisor. 2. All numbers except 1 cancelled. 3. Numbers in dividend divisible by numbers in divisor. First strike out common numbers as in first step. 4. Numbers resulting from cancellation cancelled. The order here (see below) is 6 and 3 divided by 3, 8 and 2 divided by 2, 12 and 4 divided by 4. 5. Numbers in dividend and divisor divisible by the same number. The order is 6 and 3 divided by 3, 8 and 2 divided by 2, 12 and 4 divided by 4, 15 and 10 divided by 5, 14 and 2 divided by 2. 6. Result in fractional form. 7. Numerator of result 1.

1.
$$\frac{\overset{1}{\cancel{3}} \times 5 \times \overset{1}{\cancel{2}}}{\cancel{2} \times \cancel{3}} = 5$$

2.
$$\frac{\overset{1}{\cancel{3}} \times \overset{1}{\cancel{3}} \times \overset{1}{\cancel{4}}}{\cancel{4} \times \cancel{3} \times \cancel{3}} = 1$$

3.
$$\frac{\overset{2}{\cancel{6}} \times \overset{1}{\cancel{3}} \times \overset{2}{\cancel{8}}}{\cancel{3} \times \cancel{3} \times \cancel{4}} = 4$$

4.
$$\frac{\overset{\overset{1}{\cancel{2}}}{\cancel{6}} \times \overset{3}{\cancel{12}}}{\cancel{3} \times \underset{\cancel{4}}{\cancel{8}}} = 3$$

5.
$$\frac{\overset{\overset{1}{\cancel{2}}}{\cancel{6}} \times \overset{3}{\cancel{12}} \times \overset{3}{\cancel{15}} \times \overset{7}{\cancel{14}}}{\cancel{3} \times \underset{\cancel{4}}{\cancel{8}} \times \underset{\cancel{2}}{\cancel{10}}} = 63$$

6.
$$\frac{\overset{\overset{1}{\cancel{3}}}{\cancel{9}} \times \overset{2}{\cancel{8}} \times \overset{2}{\cancel{10}}}{\cancel{3} \times \underset{\cancel{4}}{\cancel{12}} \times \underset{3}{\cancel{15}}} = \frac{4}{3}$$

7.
$$\frac{\overset{1}{\cancel{3}} \times \overset{1}{\cancel{4}} \times \overset{1}{\cancel{5}}}{\underset{2}{\cancel{6}} \times \cancel{4} \times \underset{2}{\cancel{10}}} = \frac{1}{4}$$

When possible, in the solution of problems time should be saved by cancellation. State all successive multiplications and divisions without actually performing them, then solve by cancellation. As in other work, this should be built up by easy steps.

1. If 4 hats cost $12, what will 7 hats cost? At first have problems analyzed. If 4 hats cost $12, one hat will cost one-fourth of $12 or $2. Have this placed on the board as below, emphasizing that the number above the line is the dividend and the number below, the divisor. Continue analysis. If 1 hat cost $3, 7 hats will cost 7 times $3 or $21. This will appear on the board as below before cancellation takes place.

2. If 8 caps cost $4, what will 12 caps cost? Analize and cancel as before.

3. If 12 bushels of apples cost $20, what will 9 bushels cost?

4. What will 7 turkeys cost if I pay $12 for 8 turkeys?

In later grades much more difficult problems should be solved by cancellation. In each new subject admitting the use of cancellation, practice should be given in the statement of problems. The difficulty will be found in the making of the statement, not in the cancellation.

5. How many acres in a field 80 rods by 70 rods? How shall we find the area of the field? 70 times 80 sq. rd. Indicate the operation If there are 70 times 80 square rods in the field, how shall we find the number of acres? Divide by 160 square rods. Indicate the division. Now cancel.

6. What will 914 pounds of wheat cost at 90 cents per bushel? What is the missing fact? The number of bushels. How shall we find it? Indicate this. How shall we find the cost of so many bushels? Indicate this. Cancel.

$$\frac{\$12}{4} \qquad \frac{7\times\$12}{4} \qquad \frac{12\times\$4}{8} \qquad \frac{9\times\$20}{12} \qquad \frac{7\times\$12}{8}$$

$$\frac{70\times 80 \text{ sq. rd.}}{160 \text{ sq. rd.}} = 35 \therefore 35\text{A.} \qquad \frac{914\times 90 \text{ cents.}}{60}$$

PART THREE

Denominate Numbers and Practical Measurements

I. DENOMINATE NUMBERS.

Denominate numbers should be illustrated by exact measurements. The child should have in his hands the foot rule, the yard stick, the pint, quart, and peck measures, the pint, quart, and gallon measures, and the weights to be used with the balance scales. He should discover by experiment the relation between the different measures and build up his own tables as far as practicable. He should also discover the difference between the dry and the liquid quart. He should measure and weigh various objects.

As children often have trouble in determining whether to multiply or divide in reduction of denominate numbers, the following scheme may be employed for a time.

Example: Reduce 288 inches to feet.

Write the equivalent, then write the number of inches to be reduced directly below the 12 in. As the number of feet, 1, is less than the number of inches, 12, the number of feet wanted must be less than the number of inches, 288, therefore divide.

12 in. = 1 ft.
288 in. = ? ft.
12 in. = 1 ft.
? in. = 288 ft.

Example: Reduce 288 ft. to inches.

Here evidently the number of inches must be greater than the number of feet, therefore multiply.

In this work it is important that the child be not allowed to indicate that 288 ft. multiplied by 12 gives 3456 in., nor that 288 ft. multiplied by 12 in., nor 12 in. multiplied by 288 ft. gives 3456 in. He must indicate that 12 in. multiplied by the abstract number 288 gives the desired result. 288×12 in. = 3456 in. Ans. The first example should be expressed as follows: 288 in. $\div$ 12 in. = 24 $\therefore$ 24 ft. Ans.

Example: Reduce 2 pk., 5 qt., 1 pt. to pints.

If this example is worked vertically, do not allow the work to appear as if 2 pecks multiplied by 8 equals 16 quarts, nor that 2 pints times 21 quarts equals 42 pints. Cancel the word "qt." after the 21, thus abstracting the number and consider this as the multiplier and the 2 pints as the multiplicand. If the example is solved horizontally, do not allow the following false statement: 2×8 qt. = 16 qt. + 5 qt.

$=21$ qt.; for 2 times 8 qt. equals 16 qt. only, not 16 qt. plus 5 qt. as the statement indicates. Either make several distinct statements; as, 2×8 qt. $=16$ qt. and 16 qt. $+5$ qt. $=21$ qt.; or better state as below:

8 qt.	2×8 qt. $+ 5$ qt. $= 21$ qt.
2	21×2 pt. $+ 1$ pt. $= 43$ pt. Ans.

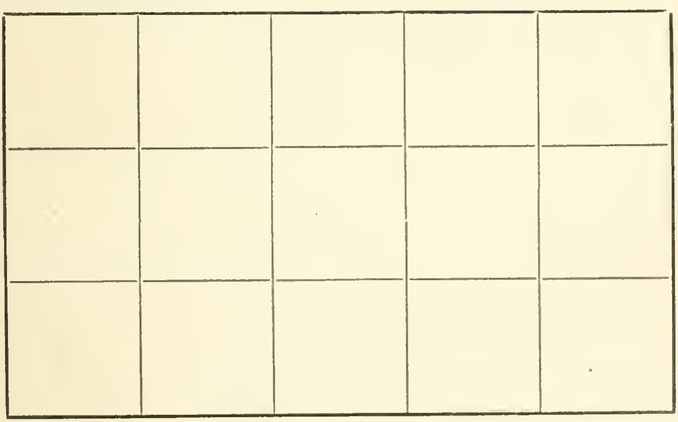
8 qt.
2
—
16 qt.
5 qt.
—
21 (qt.) Draw a line through this "qt."
2 pt.
—
42 pt.
1 pt.
—
43 pt. Ans.

In connection with the above horizontal statements, if it has not already been explained, it is well to instruct the class that it is a generally accepted mode first to perform all indicated multiplications and divisions in the order in which they occur, then to perform the indicated additions and subtractions in order. If parentheses are used, operations indicated within should first be performed.

See Walsh, pp. 129-140; Bailey, Lesson 23; Brown and Coffman, Chap. XII.

II. SQUARE MEASURE.

Example: Find the area of a surface 3 in. by 5 in.



3×5 sq. in. $= 15$ sq. in. Ans.

As there are 5 square inches in the upper row and there are 3 rows, there are 3 times 5 sq. in. in the whole surface.

In solving such problems, have the class make the illustrative diagrams until they thoroughly understand the process. Under no circumstances allow them to say 3 in. times 5 in.

Example: If a rectangle 5 inches long contains 15 square inches, how wide is the rectangle?

15 sq. in ÷ 5 sq. in. = 3.'.3 in. Ans.

If there are 5 sq. in. in one row there will be as many rows as 5 sq. in. is contained times in 15 sq. in.

In developing the number of square inches in a square foot, use the above method and have each child make a diagram of a square foot or 144 square inches on paper. Thus the square yard also should be developed and placed on the board.

III. CUBIC MEASURE.

Have several inch cubes, several boxes with the inside dimensions in exact inches if possible, and one box exactly one cubic foot inside dimensions.

Take a box, say 3 in. by 4 in. by 5 in. Place 5 inch cubes in the box in a row along one edge. Bring out that there would be 4 rows of 5 cubic inches in one layer of blocks, or 4 times 5 cu. in. or 20 cu in.; that in three such layers there would be 3 times 20 cu. in. or 60 cu. in.

$3 \times 4 \times 5$ cu. in. = 60 cu. in. Ans.

Similarly develop the number of cubic inches in one cubic foot.

IV. PRACTICAL MEASUREMENTS.

In practical measurements the operations should when possible, be reduced to a series of steps to be taken in regular order. These steps in turn may often be reduced to a set form.

V. PERIMETER-FENCES.

Bring out that Perimeter means the measure around. To find the perimeter of a rectangular figure 8 in. by 12 in., add the length of two adjacent sides: $12'' + 8''$ and multiply by 2. $2(12'' + 8'') = 40''$. the Perimeter.

Teach at once that a number before or after a parenthesis indicates that the result of the operations within the parenthesis should be multiplied by the number or numbers without.

In dictating examples in fencing, etc., have the class write the following blank on the board to be ready to take down the numbers as dictated: $2(\ \ +\ \ )$. As the example is dictated, the numbers are placed in the blank spaces. This blank will be found useful in determining the amount of fencing required to fence in a lot.

VI. AREA-ACRES.

The method of computing areas has already been given in connection with square measure.

How many acres in a farm 80 rods by 140 rods?

$$\frac{\quad\times\quad}{160} = \therefore \text{ A. Ans.} \frac{80\times 140}{160} = 70.\therefore 70 \text{ acres. Ans.}$$

Making use of the above blank, the teacher should dictate many examples for rapid computation. Insist on cencellation where possible.

VII. PLASTERING.

Take a chalk box about 8 inches by 4 inches by 4 inches. What is the perimeter of this box? Open the box out so as to make the four sides one plane surface. What is the length of the four sides? The same as the perimeter, 2(8″+4″)=24″. What is the area of the four sides? 2(8+4)4 square inches. Dictate many examples for practice in finding the areas of four sides of rooms. As the area of these four walls has been found in square feet, how shall we find the number of square yards of plastering?

Bring out the fact that the openings for doors and windows do not have to be plastered, but state that on account of the difficulty of plastering around openings plasterers generally deduct only half of the openings. Only such deductions should be made as are directed in each example. Develop the following blanks and have the class place them on paper or on the board before any computations are begun. Insist upon the placing of one equality sign directly under another.

$$\frac{2(\quad+\quad)\quad-\quad}{9} = \therefore \quad \text{sq. yd.}$$

$$\frac{\quad\times\quad}{9} = \therefore \quad \text{“}$$

$$\underline{\qquad\qquad}\ \text{“}$$

$$\times\ \$\quad = \$\quad \text{Ans.}$$

As the teacher dictates or as the child reads, he places the figures in the appropriate blanks. The computations aside from mental processes and cancellations should be performed below the line and the results extended properly.

Example: At 30 cents per square yard what will it cost to plaster a room 18′×22′×10′, allowing one-half of 100 sq. ft. for the openings?

$$\frac{2(18+22)10-50}{9} = 83\ 1/3 \text{ number of sq. yds.}$$

$$\frac{18\times 22}{9} = 44 \text{ number of sq. yds.}$$

Total 127 1/3 sq. yds.

127 1/3×\$.30 =\$38.20. Ans.

In the second statement above, cancellation should be used when possible. Give abundant practice.

VIII. CEMENT WALKS, PAVING, ETC.

Use the second statement in form for plastering. When, however, a walk is placed on two or more sides of a lot, an allowance has to be made for one or more corners; one if on two sides, two if on three sides, and four if on four sides. If the walk is on the lot, the corners are deducted; if around the outside of the lot, the corners are added. Illustrate by diagram.

Example: A. Find the number of square yards in a 4 foot cement walk on two sides of a lot 80 feet by 40 feet. B. Around two sides.

A. $4(80+40-4)\div 9=51\ 5/9$ number of sq. yd. Ans.
B. $4(80+40+4)\div 9=55\ 1/9$ number of sq. yd. Ans.

IX. PAINTING.

For rectangular rooms use the same form as in plastering. For irregular buildings find the total area in square feet before dividing by 9.

X. PAPERING.

The length of a roll of wall paper is 24 feet. Of a double roll, 48 feet. The width is generally 18 inches. What is the area of a single foll? Of a double roll? If we have the area of the surface to be covered, and know the area of a roll of paper, how shall we find the number of rolls? Is it possible to use all the paper of a roll? Some paper hangers, allowing for waste, figure upon three single rolls covering 100 square feet of surface. This is allowing about 33 square feet to the roll. Therefore divide the area to be covered by 33 square feet. For double rolls divide by 65 or 66. Salesmen of experience assert that when this method of computation is used, seldom if ever is paper returned or more called for. The strip method of computation is cumbersome and antiquated.

Example: At 75 cents per double roll what will it cost to paper a room $18' \times 22' \times 10'$, allowing 100 square feet for openings?

$$\frac{2(18+22\overset{10}{)}-100}{65}=10+\therefore 11 \text{ rolls.}$$

$$\frac{18\times 22}{65}= 6+\therefore 7 \text{ ``}$$

Total 18 rolls.

$18\times\$.75 = \$13.50.$ Ans.

If an unusual width of paper is used, a different divisor must be used.

Before beginning an example in papering, the following blank should be placed on the board or paper:

$$\frac{2(\quad + \quad) -}{33} = \therefore \text{ rolls.}$$

$$\frac{\times}{33} = \therefore \quad \text{``}$$

$$\overline{\quad\quad\text{rolls.}}$$

$$\times \ \$ \quad = \$ \qquad \text{Ans.}$$

For double rolls, substitute 65 or 66 for 33. Divisors 66⅔ and 33⅓ would be even better. To divide by 66⅔, divide by ⅔ and point off two places. To divide by 33⅓, divide by ⅓ and point off two places.

XI. CARPETING.

Though in many homes, for sanitary as well as for artistic reasons, rugs are taking the place of carpets, capets are still generally used.

PREPARATION.

If this room is 18 feet wide, how many yards wide is it? How many yard sticks placed end to end would cross the room? How many measures two feet long? 6 feet long? How do you find the number of measures each time? By dividing the width of the room by the length of the measure.

STATEMENT OF AIM.

Let us see how to find the number of strips of carpet required to cover a floor.

PRESENTATION.

Place a strip of figured wall paper on the floor. If this paper were 2 feet wide, how many strips should I lay side by side to cover the floor? If it were 3 feet wide? 6 feet wide?

COMPARISON AND ABSTRACTION.

In each case how did you find the number of strips necessary? By dividing tae width of the room by the width of the strip.

GENERALIZATION.

Then if you wished to carpet any room, how could you find the number of strips required? To find the number of strips, divide the width of the room by the width of a strip. Have children commit.

APPLICATION.

If the room is 6 yards wide and the strips ¾ yard wide, how shall we find the number of strips? Divide 6 yards by ¾ yard.

$6 \div \frac{3}{4} = 6 \times 4/3 = 8 \therefore 8$strips.

If a room is 17/3 yards wide and the carpet ¾ of a yard wide, find the number of strips.

Would a clerk cut a strip lengthwise for you? Then how many strips would you be required to buy? Give several similar examples.

Note. As carpet is purchased by the yard, it is necessary to reduce the length of the room to yards, therefore for the sake of uniformity, reduce all dimensions to yards. As the width of the room is generally reducted to an improper fraction before division by ¾, leave this dimension in the form of an improper fraction of a yard.

If a room is 19 feet wide, what must we do with the 19 feet before we divide by ¾ yard? Ans. Reduce 19 feet to yards.

$19/3$ yds. $\div \frac{3}{4}$ yds. $= 19/3 \times 4/3 = 76/9 = 8 + \therefore 9$ strips. Give similar examples.

TO FIND THE NUMBER OF YARDS.

If there are 9 strips each 8 yards long, how many yards of carpet will be required to cover the floor? Etc.

If the room is 24 feet long, what do you do to the 24 feet before multiplying? Ans. Change it to yards. Give examples.

WASTE.

To illustrate that there is generally waste on all but the first strip, use the figured wall paper.

If there is a waste of 1 foot on a strip, how many yards waste? Ans. 1/3 yard. 8″ waste? $8/36$ yd. $= 2/9$ yd. Etc.

If there are 8 strips and a waste of 10 inches on a strip, how much waste in all? $7 \times 5/18$ yds. $= 35/18$ yds. $= 1\ 17/18$ yd. Etc.

If it takes 56 yards of carpet to cover the floor and there is a waste of 2⅔ yds. in matching, how much carpet must be purchased? Etc.

ORDER OF STEPS.

I. Place on diagram all dimensions in yards.

II. Find the number of strips by dividing the width of the room by the width of one strip.

III. Find the number of yards required to cover the floor by multiplying the length of a strip by the number of strips.

IV. Find the number of yards waste by multiplying the waste on a strip by the number of strips less one.

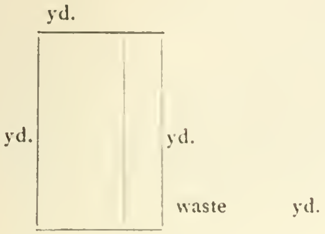
V. Find the total number of yards that must be purchased by adding the last two items.

VI. Find the cost.

Where there is no waste, omit steps IV and V.

Before dictation or work on paper begins, the following blank should be written:

I.



II. $\qquad\qquad = \quad \therefore$ strips.

III. $\quad \times$ yd. $=$ yd.

IV. $\quad \times$ yd. $=$ yd.

V. Total yd.

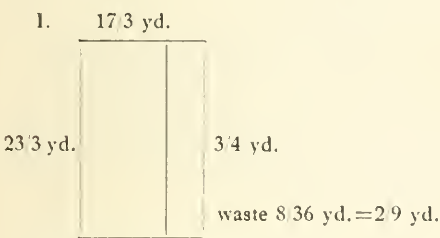
VI. $\times\$ \qquad = \$$ Ans.

Dictation should proceed slowly enough to allow the change of all dimensions to yards as the numbers are given.

Example: At $1.25 per yard what will be the cost of a 27 in. carpet for a room 17 ft. by 23 ft., allowing 8 in. waste for matching a strip?

OPERATION.

I.



II. $\frac{17}{3}\times\frac{4}{3}=\frac{68}{9} \qquad = 7+ \therefore 8$ strips.

III. $8\times\frac{23}{3}$ yd. $\qquad =61\ 1/3$ yd.

IV. $7\times\frac{2}{9}$ yd. $\qquad = 1\frac{5}{9}$ yd.

V. Total $\qquad 62\ 8/9$ yd.

VI. $62\frac{8}{9}\times\$1.25 \qquad =\$78.61.$ Ans.

XII. BOARD MEASURE.

To illustrate one board foot present two boards each $6'' \times 12'' \times 1$. Placing these side by side we have a board $1' \times 1' \times 1''$, or 1 board foot. Placing them end to end we have a board $2' \times 6'' \times 1''$ still 1 board foot. Placing one flat upon the other we have a board $1' \times 6'' \times 2''$ again 1 board foot. Thus develop rule: to find the number of board feet multiply the number of feet long by the number of feet wide by the number of inches thick, or multiply the length in feet by the width in inches by the thickness in inches and divide by 12 to reduce the width to feet. To find the number of thousand, point off three places. It is generally best not to cancel the thousand.

XIII. TO FIND THE NUMBER OF BUSHELS IN A BIN.

Approximately 1 bushel equals 5/4 cubic feet, or 1 cubic foot equals 4/5 of a bushel; hence to find the number of bushels in a bin, multiply 4/5 of a bushel by the number of cubic feet in the bin. Approximately how many bushels in a bin 10 ft. by 4 ft. by 6 ft.

$10 \times 4 \times 6 \times 4/5$ bu. $=$ 192 bu. Ans.

Accurately 2150.42 cu. in. $=$ 1 bu.; hence to find the number of bushels in a bin, reduce the number of cubic feet to cubic inches and divide by 2150.42 cu. in.

$(10 \times 4 \times 6 \times 1728$ cu. in.$) \div 2150.42$ cu. in. $= 192.85 \therefore 192.85$ bu.

XIV. TO FIND THE NUMBER OF GALLONS IN A TANK OR CISTERN.

Accurately 231 cu. in. $=$ 1 gallon. To aid in cancellation, have children commit the factors of 231, i. e. 3, 7, and 11. In all of these examples, the numbers in the curves should be placed above the line and the divisor below the line and cancellation used where possible. How many gallons in a tank 22 in. by 15 in. by 14 in.?

$(22 \times 15 \times 14) \div 231 = 20 \therefore 20$ gal. Ans.

Approximately 1 cu. ft. $= 7\frac{1}{2}$ gallons. How many gallons of water in a cistern 8 feet in diameter if the water is 7 feet deep?

$22/7 \times 4 \times 4 \times 7 \times 15/2$ gal. $=$ 2640 gal. Ans.

PART FOUR

A. Common Fractions

For suggestions regarding teaching fractions see Brown and Coffman, How to teach Arithmetic, Chap. XIII, Bailey, op. cit. Lesson 21, Walsh, op. cit. Chap. II, McLellan and Dewey, op. cit. Chap. XIII.

I. PROPER FRACTION TAUGHT OBJECTIVELY.

A. ½, ⅓, ¼, etc. B. ⅔, ¾, etc. See especially Brown and Coffman, pp. 183-4. The chief thing to be taught in class is the writing of the fraction and what is meant by the numerator and the denominator, the denominator indicating into how many parts a number has been divided, the numerator indicating how many of these parts have been taken. Paper cutting will be found as good a method of illustration as any. The terms numerator and denominator should at once be defined and used. Emphasize that the denominator is the name of the fraction, that is, the name of the parts, and tells the size of the parts or into how many parts the number has been divided. Show from the first that the larger the denominator the smaller the parts: ½ is larger than ⅓, which, in turn, is larger than ¼, etc.

II. FINDING A FRACTIONAL PART OF A NUMBER.

½ of $12, etc.

This step is given in connection with division and should dissipate the notion that a fraction merely indicates one or more parts of a single thing. Thus ½ of a number may indicate $6 as well as ½ of a dollar.

III. THE IMPROPER FRACTION.

Illustrate by piling books improperly, large upon small books.

IV. A FRACTION AN INDICATED DIVISION.

Emphasis should be laid upon the fact that a fraction is an indicated division. That the dividend is always above the line and the divisor below. Have children state which term of a given fraction is the dividend and which is the divisor. Have them write in fractional form 12 divided by three, etc. Give thorough drill till firmly fixed in mind. Here also the principle that the larger the denominator the smaller the fraction should again be emphasized and illustrated by such fractions as $12/3=4$, $12/4=3$, etc.

V. TO EXPRESS AN INTEGER AS A FRACTION.

Call on children to tell how many halves in 1, in 2, etc., using objective illustration if necessary.

1=2/2=3/3=4/4, etc. 2=4/2=6/3=8/4, etc.

VI. REDUCTION OF A FRACTION TO AN EQUIVALENT FRACTION WITH A LARGER DENOMINATOR.

Fold two equal strips of paper, one into halves, the other into fourths. Call attention to the fact that the one half is equal to two fourths and write on the board. 1/2=2/4. Similarly show that 1/2=3/6, 1/3=2/6, 2/5=4/10.

By comparison bring out that in each case both numerator and denominator may be multiplied by the same number without changing the value of the fraction. Then generalize that multiplying both numerator and denominator of a fraction by the same number does not change the value of the fraction. Give thorough drill such as: Multiply both terms of 2/3 by 4. What fraction results? What can you say of the relative value of 2/3 and 8/12? Why? Etc.

SUB LESSON—TO REDUCE A FRACTION TO AN EQUIVALENT FRACTION WITH A GIVEN DENOMINATOR.

Reduce 3/5 to an equivalent fraction with the denominator 20. 3/5=?/20. By what do you multiply 5 to obtain 20? Then if you multiply the denominator by 4, what must you do to the numerator in order not to change the value of the fraction. Give a large amount of drill in such reduction.

VII. REDUCTION TO LOWEST TERMS.

Principle to be developed: Dividing both numerator and denominator by the same number does not change the value of the fraction. Either present objectively as in the last case, or reverse the former process.

3/4=6/8∴6/8=3/4; 2/3=6/9∴6/9=2/3; 1/5=2/10∴2/10=1/5.

What do we do to 3/4 to obtain 6/8? Then what might we do to the 6/8 in order to reduce back to 3/4? Do we change the value of the fraction? Similarly for 6/9 and 2/10. Compare, generalize, and drill. Also give drill in reduction to equivalent fraction with given smaller denominator; as, reduce 8/12 to sixths. Also give drill in reduction requiring first reduction to lower then to higher terms; as, reduce 8/12 to ninths. 8/12=2/3=6/9. First reduce to lowest terms; reduce result to required fraction.

VIII. ADDITION AND SUBTRACTION OF SIMPLE FRACTIONS WITH A COMMON DENOMINATOR.

Two apples plus three apples = how many apples? Etc. Two sixths + three sixths = how many sixths? 2/6+3/6=5/6, etc.

IX. REDUCTION OF A MIXED NUMBER TO AN IMPROPER FRACTION.

Preparation—Reduction of a whole number to a fraction, and addition of fractions. Let us see how to reduce 3 2/5 to fifths. How many fifths in 3? How many fifths in 15/5 and 2/5? Then to how many fifths is 3 2/5 equal? 17/5. Similarly for 2 1/3 and 5 1/2. Compare, generalize, drill.

X. REDUCTION OF AN IMPROPER FRACTION TO A WHOLE OR TO A MIXED NUMBER.

Preparation—A fraction is an indicated division, and division with a remainder.

Presentation—Reduce 8/2 to a whole number, etc. Reduce 9/2 to a mixed number. 9/2=4 with a remainder of 1. What were the nine? Ans. Halves. Then if one of the nine halves remains, what is it? Ans. 1/2. Then 9/2=what? Ans. 4 1/2.

XI. REDUCTION TO A COMMON DENOMINATOR WHEN ONE DENOMINATOR IS A MULTIPLE OF THE OTHER AND ADDITION AND SUBTRACTION OF SUCH FRACTIONS.

How do we add three qts. and one pt.? Change 1/2 to fourths. Let us see whether we can add 3/4 and 1/2. Why not? How did we add 3 qts. and 1 pt.? To what did you say 1/2 is equal? Can you add it to 3/4 now? Similarly add 2/3 and 1/6, 1/5 and 3-10, etc. Compare, generalize, and drill. Have examples of this type added mentally as soon as possible.

XII. REDUCTION TO LEAST COMMON DENOMINATOR WHEN TWO OR MORE DENOMINATORS HAVE A COMMON FACTOR AND ADDITION AND SUBTRACTION OF SUCH FRACTIONS.

TWO FRACTIONS.

5/12+7/8. See illustration below. Divide 12 and 8 by 4, the largest number that will divide both, giving 3 and 2. Multiply the numerator and denominator of the first fraction by the 2, giving the equivalent fraction 10/24; the numerator and denominator of the second fraction by 3, giving 21/24. In this and in previous cases emphasize the idea of equivalent fractions. For a time it is well to have children draw the connecting lines as guides in multiplication.

THREE OR MORE FRACTIONS.

5/12+3/16+7/18. Teach this in the old way. See illustration below. Note that the divisor, in finding the least common denominator, should be a prime number unless it will divide all the denominators. For example, if 4 had been used in the above example instead of 2 and 2, the factor 2 would not have been eliminated from 18 and

the common denominator 288 would have been found instead of the least common denominator 144. Multiply $2 \times 2 \times 3 \times 4 \times 3 = 144$, the new denominator. Divide 144 by 12 and multiply 5 by the result, etc.

XIII. REDUCTION TO A COMMON DENOMINATOR WHEN ALL DENOMINATORS ARE PRIME TO ONE ANOTHER AND ADDITION AND SUBTRACTION OF SUCH FRACTIONS.

A. TWO FRACTIONS.

$2/3 + 1/5 = ?$ See illustration below. The ordinary method of teaching this step is to multiply the 3 by 5, then to divide the 15 by 3 to find 5 which is already before us; and to divide 15 by 5 to obtain 3. This is waste of time. Simply multiply 2 and 3 by 5, and 1 and 5 by 3. At first connect the numbers to be multiplied by lines and have the children do the same, but this should soon be dropped.

$$\frac{2}{3} + \frac{1}{5} = \frac{10}{15} + \frac{3}{15}$$

$$\begin{array}{r|rr} 4 & \frac{5}{12} + & \frac{7}{8} \\ \hline & 3 & 2 \end{array} = \frac{10}{24} + \frac{21}{24}$$

$$\begin{array}{r|rrr} 2 & \frac{5}{12} + & \frac{3}{16} + & \frac{7}{18} \\ \hline 2 & 6 & 8 & 9 \\ \hline 3 & 3 & 4 & 9 \\ \hline & 1 & 4 & 3 \end{array} = \frac{60}{144} + \frac{27}{144} + \frac{56}{144}$$

B. THREE OR MORE FRACTIONS.

$5/7 + 4/9 + 2/5$. Here instead of dividing the common denominator, 315, by 7, 9, and 5 in turn to get 45, 35, and 63, obtain these numbers by multiplying together the appropriate denominators, 9 and 5, 7 and 5, 7 and 9. The numerators of the equivalent fractions may be found thus: $5(9 \times 5) = 225$, $4(7 \times 5) = 140$, and $2(7 \times 9) = 126$.

XIV. ADDITION OF MIXED NUMBERS.

A. Sum of two fractions—1. $17\frac{2}{3} + 19\frac{3}{4} + 12\frac{1}{3}$.

Add mentally $2/3 + 1/3 = 1$, 1 plus $3/4 = 1\ 3/4$. Write $\frac{3}{4}$ and carry 1, etc. Drill should be given so that children will seek the shortest method.

B. Simple fractions that can be added mentally. $5\frac{1}{2} + 7 + 8\frac{3}{4}$, etc.

C. More difficult fractions.

See illustrations below. Before beginning work, write the word "Ans." to the right of the place where the result will finally appear. To the right of this draw a vertical line to keep the numerators of the equivalent fractions in a column. In the last example note that the denominator 6 is contained in the denominator 12, therefore find the least common denominator of 12 and 9. Divide either one of the two by 3, the largest number that will divide both, and multiply the other by the result, $9 \times (12 \div 3) = 36$, or $12 \times (9 \div 3) = 36$. As soon

as the least common denominator is found, place it below the line and to the right of the word "Ans.", leaving room above it for the sum of the numerators. Do not allow the pupils to place the mixed number 11 5/9 as equal to 20/36. They are not equal and it is a waste of time to write the 36 more than once before placing it in the answer. So, too, the numerators are more easily added if the denominators do not intervene. Reduce 5/9 to 36ths and write the numerator 20 to the right of the vertical line and opposite the numerator 5, and have the fraction read as 20/36. Similarly write the numerators 21 and 6, then adding the three numerators place the result above the common denominator 36. The children may then write the equivalent of 47/36 in its simplest form. Place the fraction 11/36 in the space reserved for the answer directly below the original fractions and carry the 1 and add the integers. As much of the work as possible should be performed mentally. If impossible to find the denominator mentally, place the denominators at one side and find the least common multiple; however, there is little value in working with difficult denominators.

19 6/7 18	11 5/9 20
12 2/3 14	16 7/12 21
17 5/7 15	9 1/6 6
5[illegible] 5/21 Ans. 47/21=2 5/21.	37 11/36 Ans. 47/36=1 11/36.

XV. SUBTRACTION OF MIXED NUMBERS.

The following seven type examples are arranged progressively. When possible have them solved mentally: 1. Nothing subtracted from a fraction. 2. One denominator divisible by the other. 3. Two denominators having a common factor. Divide 12 by the greatest common divisor 4, and multiply both 16 and 7 by the resulting 3. $3 \times 16 = 48$ (least common denominator), and $3 \times 7 = 21$. The other numerator may be similarly found: $5 \times (16 \div 4) = 20$. 4. No common factor. 5, 6, and 7 require "borrowing"; or add 1 to both minuend and subtrahend. In 7, add 1 or 35/35 to 15/35 giving 50/35, from which we subtract 28/35. Also add 1 to 401 making it 402 and then subtract.

1.	2.	3.	4.
24 3/4	17 2/3	8 7/16 21	48 2/3
7	5 5/9	5 5/12 20	17 1/2
17 3/4 Ans.	12 1/9 Ans.	3 1/48 Ans. 1/48.	31 1/6 Ans.

5.	6.	7.
		35
44	44	411 3/7 15
		50
3/5	17 3/5	401 4/5 28
43 2/5 Ans.	26 2/5 Ans.	9 22/35 Ans. 22/35.

XVI. MULTIPLICATION OF A FRACTION BY AN INTEGER.

A. Multiplication of the numerator.

2×3 apples = 6 apples, etc. 2×3 sevenths = 6 sevenths, etc. $2 \times 3/7 = 6/7$, etc. Principle generalized: Multiplying the numerator multiplies the fraction.

B. Division of the denominator.

$2 \times 3/8 = 6/8 = 3/4$, etc. Teacher erase the 6/8. What can you say of the numerator of the original fraction and the numerator of the resulting fraction? How can you obtain the 4 from the 8 and 2? Similar questioning for other fractions. Principle induced: Dividing the denominator of a fraction multiplies the fraction. Induce and have children commit thoroughly to memory: To multiply a fraction divide the denominator or multiply the numerator. Always divide if possible to save the time of reduction to lowest terms. Give much drill in multiplication of simple fractions to develop quickness in decision as to whether to divide or to multiply. This will later be a great saving of time.

Sub step.. Multiplication of a fraction by an integer that is identical with the denominator of the fraction. $4 \times 1/4 = 1$, $4 \times 3/4 = 3$, $15 \times 11/15 = 11$, etc. Have the answer written at once without writing first 3/1 and 11/1. Give much drill till the children can give results without the slightest hesitation.

C. Cancellation.

Base on earlier work in cancellation of whole numbers. $12 \times 5/16$, etc. Where the denominator is exactly divisible by the integer or the integer is exactly divisible by the denominator, have the work performed mentally without cancellation.

XVII. MULTIPLICATION OF AN INTEGER BY A FRACTION.

Teach by analysis. Find 2/3 of $6. Solve orally at first. 1/3 of $6 is $2; 2/3 of $6 is 2 times $2, (1/3 of $6), or $4. Give much oral practice, then solve on the board, $2/3 \times \$6 = \$4$, accompanying the solution with oral analysis. Rule developed: To multiply by a fraction divide by the denominator and multiply by the numerator. When the denominator is not exactly divisible, cancel if possible. The order of these operations should be determined by the relation of the integer to the terms of the fraction. In 3/4 of 8, divide first; in 3/5 of 8, multiply first. Emphasize that when 2/3 of 6 is found, 6 is multiplied by 2/3. This understanding is essential to future progress.

XVIII. MULEIPLICATION OF A FRACTION BY A FRACTION.

Use previous cancellation as preparation.

XIX. MULTIPLICATION OF A MIXED NUMBER BY A MIXED NUMBER.

This is the only case of multiplication where reduction to improper fractions is preferable. But see Bailey, **op. cit.**, p. 101.

XX. DIVISION OF A FRACTION BY AN INTEGER.

Present as in multiplication of a fraction by an integer. 15 apples÷3=5 apples. 15/16÷3=5/16. A. Principle: To divide the numerator divides the fraction. B. Principle: To multiply the denominator divides the fraction. Refer to the principle: The larger the denominator the smaller the fraction. If we make the denominator 4 times as large, how is the fraction affected? How many times smaller? Then multiplying the denominator does what to the fraction? 5/6÷4=5/24. Develop and have children commit the rule: To divide a fraction by an integer divide the numerator or multiply the denominator.

Also develop the principles: A change in the numerator produces a like change in the value of the fraction, a change in the denominator produces an opposite change in the value of the fraction. Give much practice in working examples requiring judgment as to whether it is better to divide or to multiply numerator or denominator or whether to cancel.

XXI. DIVISION OF AN INTEGER OR OF A FRACTION BY A FRACTION.

Here it is probably best, though not essential, to teach the inversion of the divisor. With young children the dictum of the teacher will doubtless be more effective than a philosophical explanation.

XXII. DIVISION OF A MIXED NUMBER BY AN INTEGER.

A. When the mixed number is small, reduce to an improper fraction and divide.

B. When the mixed number is large, divide directly and reduce the remainder to an improper fraction and complete the division.

2)1729⅔=864⅚. 2)1729⅓=864⅔. The remainder 1⅔ should mentally be reduced to 5/3 and this mentally divided by 2, which gives 5/6. 4/3 divided by 2 should at once give 2/3, not 4/6. In such examples never allow reduction to an improper fraction.

XXIII. DIVISION OF A WHOLE NUMBER BY A MIXED NUMBER.

When the divisor is not an aliquot part, multiply divisor and dividend by the denominator of the fraction and divide the resulting numbers. Basic principle: To multiply or divide both dividend and divisor by the same number does not change the quotient. This principle should be developed from the principle that multiplying both numerator and denominator of a fraction by the same number does not change the value of the fraction; and from the fact that a fraction is an indicated division.

In the example (1431 ÷ 17⅔), what can we do to both dividend and divisor without changing the quotient? Then multiply both by 3 and divide results.

At first it may be well to place in fractional form till the relation is clearly seen. Then place as below. At first allow the children to put down the multiplier, but as soon as possible have them remember that the multiplier is the same as the denominator of the fraction.

```
17⅔)1431   Multiply both by 3, placing result two lines below.

        81 Ans.
53   )4293
      424
        53
        53
```

XXIV. DIVISION OF A MIXED NUMBER BY A MIXED NUMBER

Change to improper fractions and divide.

XXV. MULTIPLICATION OF A MIXED NUMBER BY A FRACTION.

Multiply by the numerator and divide by the denominator.
4782 1/7 × 2/3. See below.

XXVI. MULTIPLICATION OF AN INTEGER BY A MIXED NUMBER.

Do not allow reduction to an improper fraction.

```
9564 2/7           434              279
————————        ——————————       ——————————
4782 1/7           217            4783 3/7
     2/3          17 2/3             93
————————        ——————————       ——————————
3188 2/21 Ans.     144 2/3          39 6/7
                  1519            14349
                  217             43047
                ——————————       ——————————
                  3833 2/3 Ans.   444858 6/7 Ans.
```

XXVII. DIVISION OF A MIXED NUMBER BY A FRACTION.

Time will generally be saved by inverting the divisor and multiplying as in multiplying a mixed number by a fraction. 28 3/7 ÷ 2/3 = ? Multiply by 3/2 as above.

At some period the following comparisons should be made: To multiply a fraction by an integer, multiply the numerator OR divide the denominator. To multiply by a fraction multiply by the numerator AND divide by the denominator.

B. Decimal Fractions

I. MEANING.

1/10=.1, 2/10=.2, 9/10=.9, etc.
45/100=.45, 1/100=.01, 7/100=.07, etc.
215/1000=.215, 9/1000=.009, 17/1000=.017, etc.

There are as many places to the right of the decimal point as there are ciphers in the denominator of the common fraction.

II. READING AND WRITING.

At first as the teacher dictates, the class should write on the board in the common fraction form and then in the decimal form. The ciphers will guide them in placing the decimal point. Give much practice in reading and writing to thousandths. Beyond thousandths "numeration" may be necessary. 0.001,786—tenths, hundredths, thousandths, ten-thousandths, hundred-thousandths, millionths—1,786 millionths. See that the -ths is annexed in all cases. Insist on correct wording when reading beyond thousandths. Do not allow the use of the word "of" in reading .4,612. Read exactly as the common fraction 4,612/10000 is read. Emphasize the use of the word "and" to indicate location of the decimal point, and the use of the decimal point to locate units. It is well to teach both of the following forms: 0.721 and .721, as both are sooner or later met with. First teach reading, then writing of many such numbers as the following: 1,000,000.000,001; 78,000,004.06,078; 317,009,000.7,048; 2,002,002.2,000,002. After multiplication and division of decimals by moving the decimal point has been taught, dictate numbers to be written, have the class divide or multiply by 10; 100; or 1,000, and write the result in words. See that such words as twenty-five and all the parts of the denominator are separated by a hyphen; as, seventy-eight million, four, and six thousand, seventy-eight hundred-thousandths.

III. PLACE VALUE.

Analyze numbers as follows. $2987.456=2000+900+80+7+.4+.5+.006$. Show that in 33.33 each 3 is ten times as great in value as the 3 to its right, and one-tenth as great in value as the 3 to its left. 3 tens or $30=10\times 3$. $3=1/10$ of 30 or of 3 tens. $3\div 10=3/10$ or .3. .3 or $3/10=10\times 3/100$ (or .03). $.03=1/10$ of .3. Etc.

IV. CIPHER AT THE RIGHT OF A DECIMAL.

Show that .20 may be read 20 hundredths and that it is equal to .2. $20/100=2/10 \therefore .20=.2$. Similarly $.2000=.200=.20=.2$, etc.

Generalization: Annexing ciphers to the right of a decimal does not change the value of the decimal. Similarly show that 1.2 may be read one and two tenths or 12 tenths. 1.2=1 2/10=12/10. Similarly 12.5=12 5/10=125/10=125 tenths. 4.73=4 73/100=473/100, etc.

V. REDUCTION OF A FRACTION TO A DECIMAL.

Preparation. A. A fraction is an indicated division. B. Division of United States money. C. 3/4=75/100=.75. Presentation. Reduce 3/4 to a decimal. As 3/4 means to divide 3 by 4, let us perform the indicated division as we divide in United States money.

$$4\overline{)\$3.00}\ \text{quotient } \$.75 \qquad 4\overline{)3.00}\ \text{quotient } .75$$

Therefore ¾=.75, the same result we found in c. Take similar examples and draw the Generalization: To reduce a fraction to a decimal, divide the numerator by the denominator and point off as in United States money.

VI. REDUCTION OF A DECIMAL TO A FRACTION.

Write the decimal as a common fraction and reduce to lowest terms. In the second example, multiply the numerator and denominator of the complex fraction by 3 and reduce the result to lowest terms.

.75=75/100=3/4. .13⅓ =13⅓/100=40/300=2/15.

VII. ADDITION AND SUBTRACTION OF DECIMALS.

Principle: Only like numbers can be added, units only can be added to units, tenths only to tenths, etc., therefore: RULE: Place one decimal point under another, keeping units under units, tenths under tenths, etc. Example, from 17 subtract .007. Emphasize that a point is understood to the right of units.

VIII. MULTIPLICATION OF DECIMALS.

Multiplication of a decimal by an integer is easily developed by referring to multiplication of United States money. Develop multiplication of a decimal by a decimal as follows: .5×.7=5/10×7/10=35/100=.35. . 3×.08=3/10×8/100=24/1000=.024. .45×.79 =45/100=79/100=3555/10000=.3555. Etc. Compare, abstract, and generalize.

IX. DIVISION OF DECIMALS.

A. Decimal by an integer.

Base on division of United States money. Place the decimal point in the result directly above the point in the dividend.

2.39	.12	1.2	.012
2)4.78	144)17.28	144)172.8	144)1.728

It is well to place the point in the result before the division begins. The first significant figure in the result should be directly above the right hand figure in the first subtrahend. All the figures should be in strictly vertical columns. The teacher should insist upon exactness in this respect, as it is essential to accuracy.

B. Division by a decimal.

Use the Austrian method. Principle: Division of both dividend and divisor by the same number does not change the quotient. See lesson on fractions, Section XXIII, for the development of this principle. Method: Move the decimal point in the divisor far enough to the right to make the divisor a whole number, move the point in the dividend the same number of places to the right, perform the division with the resulting numbers. $1.728 \div 1.44 = 172.8 \div 144$. $172.8 \div .144 = ?$ First move point in the divisor three places to the right, then move the point in the dividend three places to the right annexing sufficient ciphers, place the point in the result directly above the new point in the dividend, divide placing the first figure 1 in the result directly above the figure 2 in the dividend as the right hand 4 of the first subtrahend 144 is also placed below the 2. In dividing by .40 move the point only one place to the right and divide by short division. Divide \$120 by \$.40. In dividing by such numbers as 20, 300, 4,000, etc., move the decimal point to the left of the ciphers in the divisor, the same number of places to the left in the dividend, and divide by short division. $476 \div 400$.

1.2	1 200.	30 0.	1.19
1.44')1.72'8	.144')172.800'	.4'0)120.0'	4'00)4'76

X. FRACTION AT THE END OF A DECIMAL.

$.12\frac{1}{2} = .125$; $.4\frac{1}{8} = .41\frac{1}{4} = .412\frac{1}{2} = .4125$;

$\frac{1}{3} = .3\frac{1}{3} = .33\frac{1}{3} = .333\frac{1}{3}$, etc.

$\frac{2}{3} = .6\frac{2}{3} = .66\frac{2}{3}$, etc. $.16\frac{2}{3} = .166\frac{2}{3}$, etc.

Give sufficient practice to familiarize pupils with these equivalents, that they may be able to read and write them accurately.

XI. APPLICATION OF DECIMALS IN DIVISION BY A MIXED NUMBER.

Divide 7828 by $13\frac{1}{3}$. Multiply both numerator and denominator by 3, move the decimal points in the results one place to the left, and divide by short division.

$13\frac{1}{3}$) 7828

4.0)2348.4

587.1 Ans.

XII. ALIQUOT PARTS.

Pupils should become so familiar with the following equivalents that one immediately suggests the other. They should be encouraged to use the equivalent most economical.

16ths	12ths	11ths	10ths	9ths	8ths	7ths	6ths	5ths	4ths	3ds	half	
1/16												.06 1/4
	1/12											.08 1/3
		1/11										.09 1/11
			1/10									.10
				1/9								.11 1/9
2/16					1/8							.12 1/2
						1/7						.14 2/7
	2/12						1/6					.16 2/3
			2/10					1/5				.20
4/16	3/12				2/8				1/4			.25
			3/10									.30
	4/12			3/9			2/6			1/3		.33 1/3
6/16					3/8							.37 1/2
			4/10					2/5				.40
8/16	6/12		5/10		4/8		3/6		2/4		1/2	.50
			6/10					3/5				.60
10/16					5/8							.62 1/2
	8/12			6/9			4/6			2/3		.66 2/3
			7/10									.70
12/16	9/12				6/8				3/4			.75
			8/10					4/5				.80
	10/12						5/6					.83 1/3
14/16					7/8							.87 1/2
			9/10									.90

XIII. MULTIPLICATION BY ALIQUOT PARTS.

Considerable emphasis should be laid on this process:

A. By aliquot parts of 1. Multiply by the equivalent fraction. $.33\frac{1}{3} \times 465 = (\frac{1}{3} \times 465)$; $.14\ 2/7 \times 212 = (1/7 \times 212)$; $.66\frac{2}{3} \times 516 = \frac{2}{3} \times 516$; $.12\frac{1}{2} \times 144 = (\frac{1}{8} \times 144)$.

B. By aliquot parts of 10, 100, or 1000. Multiply by the appropriate fraction and also by 10, 100, or 1000 as the case may be; that is, multiply by the appropriate fraction and annex one, two, or three ciphers. $12\frac{1}{2} \times 144 = \frac{1}{8} \times 100 \times 144 = 1800$. $66\frac{2}{3} \times 516 = \frac{2}{3} \times 5160 = 34400$. $1.25 \times 144 = (\frac{1}{8} \times 10 \times 144) = 180$. $6.6\frac{2}{3} \times 516 = \frac{2}{3} \times 5160 = 3440$. $125 \times 144 = (\frac{1}{8} \times 1000 \times 144) = 1800$. $666\frac{2}{3} \times 516 = \frac{2}{3} \times 516{,}000 = 344{,}000$. In performing the operations, do not write down the numbers within the curves above, but compute mentally. Give much practice.

XIV. DIVISION BY ALIQUOT PARTS.

A. By aliquot parts of 1. Divide by the equivalent fraction. $252 \div .33\frac{1}{3} = (252 \div \frac{1}{3}) = 756$. $144 \div 12\frac{1}{2} = (144 \div \frac{1}{8}) = 1152$. $516 \div .66\frac{2}{3} = 516 \times 3/2 = 774$.

B. By aliquot parts of 10, 100, or 1000. Divide by the appropriate fraction and by 10, 100, or 1000; that is, divide by the appropriate fraction and point off one, two, or three places.

$1779 \div 16\frac{2}{3} = (1779 \div \frac{1}{6} \div 100) = 106.74$.
$279 \div 333\frac{1}{3} = (279 \div \frac{1}{3} \div 1000) = .837$.
$7421 \div 3\frac{1}{3} = 2226.3$ $144 \div 12\frac{1}{2} = (144 \div \frac{1}{8} \div 100) = 11.52$.
$516 \div 66\frac{2}{3} = 5.16 \times 3/2 = 7.74$. $144 \div 1.25 = 11.52$.
$516 \div 6.66\frac{2}{3} = 51.6 \times 3/2 = 77.4$. $144 \div 125 = 1.152$.
$516 \div 666\frac{2}{3} = .516 \times 3/2 = .774$.

C. Fractional Relations

I. TO FIND A FRACTIONAL PART OF A NUMBER.

Find $\frac{2}{3}$ of \$18. Find .7 of \$250. Find .17 of \$500.
Fraction × whole = part.
Factor × factor = product.
$\frac{2}{3} \times \$18 = \$12$.
$.7 \times \$250 = \$175$.
$.17 \times \$500 = \$85.00$.

II. TO FIND WHAT PART ONE NUMBER IS OF ANOTHER.

Preparation:

Factor		Factor		Product
2	×	3	=	6

Given the product 6 and one factor 3, how do we find the other factor? In general, how do we find the other factor when one factor and the produce are given?

Ans. Divide the product by the given factor to find the other factor.

Factor		Factor		Product
⅔	×	$18	=	$12.
.7	×	$250	=	$175.
.17	×	$500	=	$85.00.

Statement of aim: Let us see how to find what part one number is of another. $12 is what part of $18? $175 is what part of $250? $85 is what part of $500?

Presentation:

Statement of relation:

Factor		Factor		Product
?	×	$18	=	$12.
?	×	$250	=	$175.
?	×	$500	=	$85.

Given the product $12 and the factor $18, how shall we find the other factor? The product $175 and the factor $250? Etc.

Ans. Divide the product $12 by the factor $18. Divide the product $175 by the factor $250. Etc.

Statement for solution: 12/18=2/3 Ans. $175÷$250=.7 Ans. $85÷$500=.17 Ans.

In finding what part one number is of another it is some times preferable to place the division in fractional form and reduce to lowest terms as above.

The two satements, the statement of relation and the statement for solution may be used whenever these examples are to be worked; or the teacher may develop the rule to divide the part by the whole to find the fraction.

III. TO FIND THE WHOLE WHEN A PART AND ITS FRACTIONAL RELATION TO THE WHOLE ARE GIVEN.

Preparation: The product divided by the factor gives the other factor.

Factor		Factor		Product
⅔	×	$18	=	$12.
.17	×	$500	=	$85.
.7	×	$250	=	$175.

Statement of aim: Let us see how to find the whole when the part and its relation to the whole are given. $12 is ⅔ of how many dollars?

Presentation.

Statement of relation:

Factor		Factor		Product
⅔	×	$?	=	$12.
.7	×	$?	=	$175.
.17	×	$?	=	$85.

Statement for solution:

$12 ÷ ⅔ = $ Ans.

$175 ÷ .7 = $ Ans.

$85 ÷ .17 = $ Ans.

As in the preceeding case, the two statements may be used or the teacher may develop the rule to divide the part by the fraction to find the whole. Call attention to the fact that when "of" is used to indicate multiplication, the whole always follows the word "of". Give an abundance of drill on these three cases of fractional relations as a preparation for Percentage. Have the children thoroughly understand that when the part is wanted they always multiply, hence the **part** is the **product**. When the **part** is given it is always divided by whichever factor is given to find the other factor. At first the statements of relation and solution should both be written before the example is worked, a blank space being left for the insertion of the answer.

.7 × $? = $17.50.	? × $18 = $12.
$175 ÷ .7 = $ Ans.	$12/$18 = 2/3. Ans.

D. Oral Analysis

The old method of oral analysis should be used in the solution of simple problems, though it should not be allowed to crowd out the work in **fractional relations** which is a preparation for Percentage.

Example: Find ¾ of $1200. Solution: ¼ of $1200 is $300, and ¾ of $1200 is 3 times $300 or $900.

Example: $900 is ¾ of how many dollars? Solution: ¼ of the money (which is ⅓ of ¾ of the money), is ⅓ of $900 or $300, and 4/4 of the money is 4 times $300 or $1200.

As oral work should be given daily, a class should soon become proficient in analysis. The statement within the curves is explanatory and should not be required in the solution, though the children should be able to give it if requested.

PART FIVE

Percentage

I. PREPARATION FOR PERCENTAGE.

As a preparation for Percentage, aliquot parts and fractional relations should be thoroughly mastered. Much use should be made of fractions and decimals with denominators of 100 and of the equivalent fraction in its lowest terms. While decimals are a special case in fractions with denominators of ten or powers of ten, Percentage is a special case in decimals with denominator 100. This should be made clear to the class. The following questions will illustrate the method of procedure:

12 is what part of 16? How many hundredths? write the fraction as a decimal. What part of 15 is 10? How many hundredths? What part of 7 is 6? Change to the decimal form expressing it as hundredths. Etc.

Find .75 of $84. Find ¾ of $84. Compare the results. Find .25 of $738. Find ¼ of $738. Compare results. Find .66⅔ of $18. To what fraction is .66⅔ equal? Find ⅔ of $18. Compare results. Which is the shorter way? Find .16⅔ of $72. By what simple fraction may we multiply and obtain the same result? Find .07 of $17. Is there any simple fractional equivalent of .07? Then what shall we multiply by in this case? Etc.

$15 is .05 of how many dollars? $18 is .66⅔ of how many dollars? What simple fraction may we substitute for the decimal? Do it and solve. Etc.

Drill until the children can readily solve simple problems in decimals and fractions with denominators of 100, using the decimal when there is no simple fractional equivalent, and exchanging fractions and decimals without hesitation when it is advantageous to do so. When the class understand the precess, tell them that they have been working examples in "**Percentage**", a name applied to examples in fractions and decimals in which the denominator is 100.

II. INTRODUCTION OF THE TERMS PER CENT, RATE PER CENT, AND RATE.

As soon as the children are familiar with the method of working examples in hundredths, the terms **per cent**, **rate per cent**, and rate should be introduced. The class should become familiar with these terms before the introduction of the term **percentage** that the two terms may not be confused as sometimes happens when they are introduced together. Tell the children that as the term per cent is used

it means exactly the same as the word hundredths. Also teach the symbol for per cent and have the class write the various equivalents.

75 per cent = 75% = .75 = 75/100 = ¾.

62½ per cent = 62½% = .62½ = .625 = ⅝.

7 per cent = 7% = .07 = 7/100. Etc.

Emphasize that as per cent means hundredths, when a per cent is written as a decimal, two decimal places are necessary. Give much practice in writing the per cent as a decimal of two places, then have the fractional part of the per cent also written as a decimal. Read ½%, ¼%, etc., as ½ of one per cent, ¼ of one per cent, etc.

62½% = .62½ = .625. 7¼% = .07¼ = .0725.

½% = .00½ = .005. ¼% = .00¼ = .0025.

⅓% = .00⅓ = .0033⅓ = .003333⅓, etc.

Children should also acquire facility in changing such decimals to the per cent form. To demonstrate to the class the decimal equivalent of the fractional part of a per cent, use the following subtraction:

(a) 17½% = .17½ = .175.

(b) 17% = .17 = .17.

(c) ½% = .00½ = .005. Subtracting b from a.

III. INTRODUCTION OF THE TERMS PERCENTAGE AND BASE

Review the words whole and part as used in fractional relations and tell the class that in the study of Percentage new terms are used for the words part and whole. For the whole we use the term Base, and for the part, the term Percentage. Write on the board:

$8 is 25% of $32. $8 is ¼ of $32. $8 is 25% of what quantity? Find 25% of $32. $8 is what per cent of $32?

Ask such questions as: What is the rate? What is the whole? Then what shall we call this in percentage? What is the part? What is the new word that we have learned for part? Or in other examples: Can you tell which is the percentage? If not, ask yourself which is the part? Can you tell now? Which is the base? Or: What do we call the 25%? What is the $32? Is it the part or the whole? Then what shall we call it? What is called for? Etc. Continue such questioning till the children can tell at once which term to apply to each number without first asking themselves which is the part, etc. It is not necessary to have these examples worked as the drill is in the application of terms. It is important that the class understand the terms and can apply them before they try to solve problems.

IV. GIVEN THE BASE AND RATE TO FIND THE PERCENTAGE

Preparation:

Find .17 of $59. Find .73 of $714. Find .08 of $216. What new

way have we learned to express .17? Ans. 17% Similar questions for .73 and .08. What do you call the 17%, 73%, and 8%? The Rate per cent. What do we call the $59, $714, and $216? The Base.

Sub step—Statement of aim.

Let us see now how to find the Percentage when the Rate and Base are given.

Presentation:

Let us find 17% of $59. What did you say the $59 is? The 17%? What did we find was the equivalent of 17%? And how did you find .17 of $59? Then how shall we find 17% of $59? Ans. Multiply .17 times $59. Do it. What do we call the result? Ans. The Percentage. How did we get it? Ans. We multiplied .17 times $59. But what is the 17% that you multiplied by? The $59? Then how did you get the Percentage? Ans. We multiplied the Rate times the Base. Similarly work and question regarding two other examples.

Comparison.

How did you find the Percentage in each of the three examples you worked? We multiplied the Rate times the Base.

Generalization:

Who will give me a rule for working all examples when the Rate and Base are given and we wish to find the Percentage? Rule: To find the Percentage when the Rate and Base are given, multiply the Rate times the Base. If we let R stand for Rate, B for Base, and P for Percentage, who will write this rule as a **formula.**

$$R \times B = P.$$

This we shall call the Formula for Percentage and shall read it: **The Rate times the Base equals the Percentage.**

Application:

Send class to board to write formula and interpret it. Each member of the class should leave the formula written at the top of the board and as the teacher dictates examples to be worked, the numbers should be written below the appropriate letter in complete statement form. Example: The Base is $218, the Rate 15%, find the Percentage. Each number is properly placed as dictation proceeds and the following should appear on the board before work begins:

$$\begin{array}{ccccccc} R & \times & B & = & P. & & \\ .15 & \times & \$218 & = & \$ & \qquad & \text{Ans.} \end{array}$$

Insist on complete statement before work begins. The example should then be worked below and the result placed in the blank left in the statement.

Assignment:

Problems should be read from the text book, the numbers identified by the class in terms of percentage, and the method of solution stated. The probliems should then be worked during the study period

V. GIVEN THE BASE AND THE PERCENTAGE TO FIND THE RATE.

Review fractional relations to find what part one number is of another. Place formula on the board and below its appropriate letter each number of the example. $12 is what per cent of $36? The class should write the Formula, R×B=P, and below it the Statement of relation, ?×$36=$12. As the Percentage is the result of multiplying the Rate and the Base, what is the Percentage of the Rate and Base? Ans. The product. Then given the product $12 and the factor $36, how shall we find the other factor, the Rate? State it. The example should now appear on the board as follows:

(Formula)	R	×	B	=	P
(Statement of relation)	?	×	$36	=	$12
(Statement for solution)	$12/$36 =1/3=33 1/3% Ans.				

Example: $28.63 is what per cent of $409?

R	×	B	=	P
R	×	$409	=	$28.63
$28.63÷ $409			=	% Ans.

Solve and place the result in the blank. For a time give simple examples and require the use of the formula. Then give problems requiring identification of data. In time the class should make the statement for solution as soon as the data has been identified.

VI. GIVEN THE PERCENTAGE AND THE RATE TO FIND THE BASE.

Example: The Percentage is $35, the Rate 5%, find the Base.

Formula	R	×	B	=	P	
Statement of relation	.05	×	$B	=	$35	
Statement for solution	$35	÷	.05	=	$	Ans.

After giving the three cases in Percentage, emphasize the fact that when the Percentage is wanted, we multiply; when the Percentage is given, it is divided by the given factor. Dictate many examples merely to be stated on the board without being worked. Dictation should be rapid, but not so rapid but that all members of the class can write the first number before another number is given. Unless it is the teacher's fault, do not repeat a number. See that every member of the class writes every number as it leaves the teacher's lips. If a child will not keep up, send him to his seat. He will soon learn not to lag behind. It is an inexcusable waste of time on the part of the teacher to repeat for an inattentive pupil. Such repetition begets inattention.

Do not allow the pupils to hold the erasers in hand and erase when they please. Leave all the work on the board until the spaces are well filled, then say "erase" and have all erase at the same time, thus saving valuable time.

Many examples may be given to be stated without solving. The formula and statement of relation should be written and the statement for solution when this differs from the statement of relation. The blank should always be left for the answer. Many simple oral problems should be given to the class at their seats, also many difficult problems requiring identification of data should be read from the text book, the numbers identified in terms of percentage, and the method of solution indicated orally.

A point that should be brought out at this time is that as one whole may be divided into several parts, so one Base may have several Percentages and for each Percentage there is a rate.

Example: Mr. Knox had $190 and spent $40 for a suit, $35 for an overcoat, $5 for a hat, and $7 for a pair of shoes. How much did he have left? What per cent had he left? What per cent did the suit cost? The overcoat? Etc. What per cent did he spend altogether? Prove this result by adding the per cents spent for the several items. Also add the per cents spent to the per cent left, the sum being 100%.

Example: Mr. Patterson had a salary of $150 per month. He spent 20% for rent, 8⅓% for insurance, 26⅔% for household expenses, 33⅓% for sundries, and saved the rest. What per cent did he save? What was his rent? Etc. Check: the sum of the several Percentages should equal the Base.

VII. AMOUNT.

James paid 40 cents for some papers and sold them at a gain of 20 cents. What did he sell them for? This we call the Amount. What is the 40 cents in terms of percentage? The 20 cents? And what is the 60 cents called? Then how did you find the Amount? Who will define Amount? The Base plus the Percentage equals the Amount. We will write this as follows: $B+P=A$, and call it the definition of Amount. Have children commit to memory. Give many examples to find the Amount when the Base and Percentage are given. Especially emphasize that the selling price when there is a gain is the Amount.

When we have given the sum of two numbers and one of the numbers, how do we find the other number? If 12 is the sum of 8 and some other number, what is the other number? Mr. Ackerson paid $150 for a horse and sold it for $180, what did he gain? What do we call the cost? The selling price when there is a gain? The gain? (The teacher should emphasize that the gain is always the Percentage.) Then how did we find the Percentage when the Base and the Amount were given? Give several examples.

Mr. Hughes sold a watch for $30, thereby gaining $10, What did the watch cost? What was the $30? The $10? The 20? Then how did you find the Base when the Amount and Percentage were

given? Give many examples illustrating the three kinds of examples till the class can work them without hesitation.

VIII. GIVEN THE AMOUNT AND THE BASE TO FIND THE RATE.

Emphasize that the Rate is always found by dividing the Percentage by the Base, therefore, to find the Rate when the Base and Amount are given, find the missing fact, the Percentage, and use the formula for Percentage. This should be developed by means of questions.

IX. GIVEN THE AMOUNT AND THE PERCENTAGE TO FIND THE RATE.

By questioning, develop: To find the Rate when the Amount and Percentage are given, find the missing fact, the Base, and use the formula for Percentage. Give many examples illustrating the last two cases.

X. GIVEN THE RATE AND THE BASE TO FIND THE AMOUNT.

To find the Amount when the Rate and the Base are given, at first have the class solve by finding the missing fact, the Percentage, then add this to the Base. But it is an advantage to be able to solve by multiplying by 1 plus the Rate, also it is essential to an understanding of the next case. Before teaching the following lesson, the fact should be emphasized that any number is 100% of itself. \$20 is 100% of \$20. 100% of \$75 is \$75. The purpose of the following lesson is to teach the formula: $(1.00+R)\times B=A$.

Preparation:

$R\times B=P$. The Base equals 100 of itself. Base plus Percentage equals the Amount.

$R\times B=P$.

$.05\times \$60=\$3$. $.06\times \$50=\$3$. $.07\times \$80=\$5.60$.

$\$60+\$3=\$63$. A. $\$50+\$3=\$53$ A. $\$80+\$5.60=\$85.60$ A.

Statement of aim: We are now going to learn a better way of finding the Amount when the Rate and the Base are given.

Presentation:

\$60 is what per cent of \$60? Ans. 100% of \$60. Teacher write on board: 100% of \$60 = \$60. \$3 is what per cent of \$60? Teacher write on board below last statement: 5% of \$60 = \$3. How many per cent of \$60 is the sum of 100% of \$60 and 5% of \$60? Ans. 105% of \$60. Teacher write on board below the other statements. How many dollars is the sum of 100% of \$60 and 5% of \$60? Ans. \$63. Teacher write on board. We now have the following:

	100%	of \$60 =	\$60
	5%	of \$60 =	\$ 3
	----		----
Adding	105%	of \$60 =	\$63

Send child to board to verify this statement by multiplying $60 by 1.05. What is the 5%? Ans. The Rate. What is the 105%. Ans. 100% plus 5% or 100% plus the Rate. What is the $60? The $63? Then how did we find the Amount? or What did we multiply the Base by to find the Amount? Ans. 100% plus the Rate. Similarly treat the other two examples used in the preparation.

Comparison:

In all these examples how did we find the Amount? Ans. We multiplied 100% plus the Rate times the Base.

Generalization:

Give me a rule for finding the Amount when the Rate and the Base are given. Rule: To find the Amount when the Rate and the Base are given, multiply the Base by 100% plus the Rate. We may write this as a formula thus: $(1.00+R)\times B=A$.

This we shall call the **formula for Amount.** It may be read either 100% plus the Rate times the Base equals the Amount, or 1 plus the Rate times the Base equals the Amount. The use of the curves in writing the formula is necessary. See that the class differentiate between the formula and the definition.

Application:

Send class to board to write and interpret the formula. Then dictate examples to be written below the formula in statements. At first have examples stated only, then have some of them worked. Example: The Base is $246, the Rate is 8%, state for Amount.

$$(1.00+R)\times B=A.$$
$$1.08\times \$246=\$ \qquad \text{Ans.}$$

The addition of 1.00 and the Rate should always be made mentally and the example stated as above at once as the teacher dictates.

Assignment:

Problems from the text book should be read and the numbers identified in terms of percentage and the method of solution indicated. They should then be worked during the study period.

XI. GIVEN THE AMOUNT AND THE RATE TO FIND THE BASE

Example: The Amount is $428, the Rate 7%, state for finding the Base.

Formula	$(1.00+R)\times B=A$	
Statement of relation	$1.07\times \$B=\$428$	
Statement for solution	$\$428\div 1.07=\$$	Ans.

Upon reaching this point the majority of a class will work this example without help from the teacher except the suggestion that the formula for Amount be used and the numbers placed under the appropriate letters.

XII. GIVEN THE AMOUNT AND THE RATE TO FIND THE PERCENTAGE.

Develop: Find the essential missing fact, the Base, and subtract.

XIII. GIVEN THE PERCENTAGE AND RATE TO FIND THE AMOUNT.

Develop: Find the essential missing fact, the Base, and add.

Many miscellaneous problems applying all the cases in percentage thus far covered should now be given for identification of data, for statement, or for solution. Also give frequent special drills in identification of data as follows: Send class to the board to write the appropriate letter as the teacher dictates; the whole, the part, the cost, the gain, the selling price when there is a gain, what you have to start with, etc. Teacher should dictate rapidly and then have all errors checked. The order of dictation should vary.

XIV. DIFFERENCE.

A dealer paid $12 for some goods and then on account of their being damaged sold them at a loss of $3, what was the selling price? What is the $12? The $3? Then how did you find the $9? Ans. We subtracted the Percentage from the Base. We call this $9 the Difference because it is obtained by subtraction. Define Difference. The Base minus the Percentage equals the Difference. We write this thus: $B - P = D$, and call it the definition of Difference. If the Base is $74 and the Percentage $14, what is the Difference? Etc. Emphasis should be laid on the fact that the selling price when there is a loss is the Difference.

Mr. Jerome paid $35 for vegetables and sold them for $30, how much did he lose? What is the $35? The $30? The $5? Then how did you find the Percentage when the Base and the Difference were given? Etc.

Frank sold his sister a tablet for $.10 thereby losing $.05, what did the tablet cost him? What is the $.10? The $.05? The $.15? Then how did you find the Base when the Difference and Percentage were given? Etc. The class should become proficient in the three kinds of examples given above.

XV. GIVEN THE DIFFERENCE AND THE BASE TO FIND THE RATE.

Develop: Find the essential missing fact, the Percentage, then find the Rate in the usual way.

XVI. GIVEN THE DIFFERENCE AND THE PERCENTAGE TO FIND THE RATE.

Develop: Find the essential missing fact, the Base, again em-

phasizing that the Percentage is always divided by the Base to find the Rate.

XVII. GIVEN THE RATE AND THE BASE TO FIND THE DIFFERENCE.

To find the Difference when the Rate and the Base are given, at first have the class solve by finding the missing fact, the Percentage, then subtract this from the Base. Later teach the following formula: $(1.00-R)\times B=D$. See that the class differentiate between the formula and the definition.

This lesson should be presented as the corresponding lesson in Amount was presented. The examples in the preparation would be like the following:

$$R\times B=P$$
$$.05\times \$60=\$3$$
$$\$60-\$3=\$57 \text{ D}$$

The questions in the presentation should lead to the following:

$$100\% \text{ of } \$60=\$60$$
$$5\% \text{ of } \$60=\$3$$

Subtracting, 95% of $\$60=\$57$.

Verify and bring out that to find the Difference we multiplied the Base by 100% minus the Rate. In the application have class write formula and statement, subtracting the Rate from 1.00 mentally. Example: The Base is $380, the Rate 12%, fiind the Difference.

$$(1.00-R)\times B=D$$
$$.88\times \$380=\$\qquad \text{Ans.}$$

XVIII. GIVEN THE DIFFERENCE AND THE RATE TO FIND THE BASE.

Example: The Difference is $399.50, the Rate 15%, state for finding the Base.

Formula	$(1.00-R)\times B=D$	
Statement of relation	$.85\times \$B=\$399.50$	
Statement of solution	$\$399.50\div .83=\$$	Ans.

XIX. GIVEN THE DIFFERENCE AND THE RATE TO FIND THE PERCENTAGE.

Develop: Find the essential missing fact, the Base and subtract.

XX. GIVEN THE PERCENTAGE AND THE RATE TO FIND THE DIFFERENCE.

Find the essential missing fact, the Base, and subtract.

Examples in Profit and Loss, Commission, and Cash Discount should be given to apply all the cases in percentage. Continue the drill in identification by writing the appropriate letter on the board as the teacher dictates. Now the terms dictated may include: what you start with, the whole, the cost, the list or marked price, the selling price in commission, the principal, all of which are the Base; the part, the gain, the loss, the commission, the discount, all of which are the Percentage; the selling price when there is a gain, the sum sent to an agent to pay for goods and commission, any sum after a gain, Amount; the part left, the selling price when there is a loss, the selling price when there is a discount, the net proceeds of a sale on commission when there are no extra expenses, Difference. When the Rate is called for, have class state what Rate; as, Rate of commission, Rate of gain, etc. Emphasize that the word after "of" following "Rate" indicates the Percentage. When the Rate is the Rate of commission, the Percentage is the commission; when the Percentage is the gain, the Rate is the Rate of gain; etc. Emphasize that the selling price may be either Base, Amount, or Difference, bringing out that it must be known whether goods are sold at a loss or a gain, on commission, or at a discount, before the selling price can be identified. Also after calling attention to the fact that in profit and loss the cost is always the Base, bring out that in discount the Difference, which is the selling price to the man selling the goods, is the cost to the man buying.

In connection with the work in percentage, the teacher should review the section on "Problem Solving"; and when the class are writing down the identification of data either on paper or the board, they should always indicate what Rate is given or wanted; as, the "Rate of loss" or the "Rate of Comm.", not merely the "Rate". This often is a material aid in identifying the Percentage itself.

XXI. SEVERAL SUCCESSIVE DISCOUNTS.

Example: The list price of a piano sold by Potter and Putnam is $600 with 10%, 8%, and 5% off. What is the selling price?

The example may be worked in either of two ways. Ordinarily the quickest way is to use the following formula:

$$(1.00-R_1)(1.00-R_2)(1.00-R_3)\text{ etc.})\times B = S.P.$$

The child should subtract the Rates mentally, stating the example at once:

$$.90\times.92\times.95\times\$600=\$471.96.\quad\text{Ans.}$$

This method certainly has the advantage of compactness of statement, while in the number of figures required in the operations there is also a saving. The child working at the board to the teacher's dictation can state and work the example by this method in half the time required by the method commonly in use. As the teacher dictates, the child makes the mental subtractions and has the example

stated as soon as dictation ceases. All that remains to be done is to make the abstract computations and place the answer in the statement. The constant oscillation between statement and work is avoided. Give much practice in merely stating such problems without solving them.

XXII. TO FIND A SINGLE RATE OF DISCOUNT EQUIVALENT TO SEVERAL SUCCESSIVE RATES.

Use the following formula:

$$1.00-(1.00-R_1)\,(1.00-R_2)\,(\text{etc.})=R$$

Example: What single Rate of discount is equivalent to 8% and 10% off?

$$1.00-.92\times.90=17\tfrac{1}{5}\%.\quad \text{Ans.}$$

XXIII. MARKING GOODS.

Merchants sometimes wish to mark goods for a sale so that they may be able to make a profit on the cost of the goods but at the same time be able to offer a discount on the marked price. For example, a dealer wishes to sell goods that cost $240 so that he will make a profit of 25% on this cost, but still be able to discount his marked price 20%. To make 25%, he must sell for $300. He marks his goods $375 and sells at a discount of 20% on this marked price, thus receiving the $300 he wishes. This involves a change of Base, the Base being the cost in profit and loss, and the marked price in discount. The selling price is the Amount in profit, and the Difference in discount, therefore the $300 is the Amount in one case and the Difference in the other. We multiply the Base $240 by 1.25 to obtain the selling price, then divide this selling price as the Difference in discount by 1.00—R, or .80, giving us the other Base, the marked price. That is, $(1.00+R_1)\times B$ is the selling price, R_1 being the Rate of gain and B the cost. This selling price as the Difference divided by $(1.00-R_2)$ gives us the other Base, the marked price, R_2 being the Rate of discount. Expressing this as a formula, we have:

$$\frac{(1.00+R_1)\times B}{1.00-R_2} = \text{Marked Price.}$$

As the teacher dictates the problem, the class should add and subtract the Rates mentally and state the problem promptly as follows:

$$\frac{1.25\times\$300}{.80} = \$ \qquad \text{Ans.}$$

As usual the teacher should give problems to be stated only, sometimes giving the Rate of discount before the Rate of gain.

If the cost is not stated, and the question is how must goods be marked to gain 25% and still allow a discout of 20%? assume $1.00 as the cost.

$$\frac{1.25\times\$1}{.80}=\$1.56\ 1/4.\ \therefore .56\ 1/4\%\ \text{above cost.}$$

But as 1 as a factor does not change the result, the $1 may be omitted and the example stated as follows:

$$\frac{1.25}{.80}=1.56\ 1/4.\ \therefore .56\ 1/4\%\ \text{above cost.}$$

Put as a formula this would be either:

$$\frac{1.00+R_1}{1.00-R_2}=\text{M. P. in per cent.}$$

Or if the formula is to express the Rate above cost:

$$\frac{1.00+R_1}{1.00-R_2}-1=\text{R above cost.}$$

Example: How must pianos costing $850, $500, and $300 be marked so that the dealer may give a discount of 20% and still gain 25%?

1.25/.80=1.56¼. 1.56¼ × $500 = $ Ans. 1.56¼ × $850 = $ Ans. 1.56¼ × $300 = $ Ans. It will be seen that the advantage of this method is that the dealer gets his Rate once for all articles, and then multiplies each cost by this Rate, instead of multiplying the cost of each article by 1.25 and dividing each result by .80.

XXIV. SIMPLE INTEREST.

As a preparation for Simple Interest, teach reduction of months and days to years using the following progressive order: months to years; years and months to years; days to years; years and days to years; months to days; months and days to days and then to years; years, months and days to years, first reducing the months and days to days. Four months equals 4/12 of a year equals 1/3 of a year; 3 years, 4 months equals 10/3 years; 20 days equals 20/360 of a year equals 1/18 of a year; 4 years, 18 days equals 4 1/20 years equals 81/20 years; 3 months, 10 days equals 100 days equals 5/18 of a year; 2 years, 3 months, 10 days equals 2 5/18 years equals 41/18 years.

Interest, which is money paid for the use of mony or capital, is paid annually, hence we have a new element, Time. What will be the Interest for one year on $450 at 5%?

$$R\times\ B=P$$
$$.05\times\$450=\$22.50\ \text{Interest.}$$

What will be the Interest for 2 years? 3 years? ½ year?

$2 \times .05 \times \$450.00 = \$45.00$ Interest for 2 years.
$3 \times .05 \times \$450.00 = \$67.50$ Interest for 3 years.
$1/2 \times .05 \times \$450.00 = \$11.25$ Interest for 1/2 year.

Letting I stand for Interest and T for Time in years, we have in general: $T \times R \times B = I$. What will be the Interest on $300 for 3 yrs. 4 mos. at 6%? (Use cancellation.)

$T \times R \times B = I$
$10/3 \times .06 \times \$300 = \$60$. Ans.

What will be the Interest on $600 for 2 years, 12 days at 5%?

In computing Interest, one year is now generally treated as 360 days, and one month as 30 days; therefore 12 days equals 12/360 of a year, or 1/30 yr.

$61/30 \times .05 \times \$600 = \$61$. Ans.

Find the Interest on $200 for 9 mo. 10 da. at 3%.

$280/360 = 7/9$. $7/9 \times .03 \times \$200 = \$4.66\frac{2}{3} \therefore \$4.67$. Ans.

Find the Interest on $115 for 2 yrs., 1 mo., 10 da., at 5%.

$19/9 \times .05 \times \$115 = \$12.14$. Ans.

XXV. GIVEN THE INTEREST AND TWO ELEMENTS TO FIND THE THIRD ELEMENT.

Preparation: $2 \times 3 \times 5 = 30$. $30 \div (2 \times 5) = 3$; $30 \div (2 \times 3) = 5$; $30 \div (3 \times 5) = 2$. $T \times R \times B = I$.

Example: In how long a time will $470 placed at 4% Interest produce $65.80 Interest?

$T \times .04 \times \$470 = \$65.80$.
$\$65.80 \div (.04 \times \$470) = 3\frac{1}{2} \therefore 3$ yr., 6 mo. Ans.

If the result were 2 7/9 years, 7/9 of 360 days is 280 days which equals 9 months 10 days, giving 2 years, 9 months, 10 days.

Example: At what rate will $240 produce $32.40 interest in 2 years, 3 months?

$\$32.40 \div (2\frac{1}{4} \times \$240) = .06$. $\therefore 6\%$ Ans.

Example: What principle placed at 5% interest for 4 years will produce $28.80 interest?

$\$28.80 \div (4 \times .05) = \$144$. Ans.

XXVI. SIX PER CENT METHOD.

The teacher should develop the method of subtracting dates and of finding the Rate at 6% for the given time. For one month the Rate would be one twelfth of .06 or .005; for one day one thirtieth of .005 or .000⅙ or .001/6.

I. Find the Time by subtracting dates if necessary.

II. Find the Total Rate for the given time:

$$
\begin{array}{lll}
\text{Number of years} & \times .06 & = . \\
\text{Number of months} & \times .005 & = . \\
\text{Number of days} & \times .000\tfrac{1}{6} & = . \\
\hline
\text{Total Rate} & & .
\end{array}
$$

III. Find the Interest:

$r/6 \times \text{Total } R \times B = I.$

IV. Find the Amount:

$B + I = A.$

Multiplying the principal by the Total Rate for the given time would give the Interest at 6%. Dividing this by 6 gives the Interest at 1%; and multiplying the Interest at 1% by r (not by R), gives the Interest at the given Rate. Here r equals the number of per cent. For example: if the Rate is 5%, $R = .05$ and $r = 5$. In practice we do not actually find the Interest at 6% and then divide by 6 and multiply by r. We at once cancel the 6 when possible and then multiply in any order deemed shortest. Have the class place blank on the board before beginning dictation. Insist on the vertical arrangement and correct relative position of all work by all members of class. In the end this is a time saving device.

SIX PER CENT BLANK.

$$
\begin{array}{lll}
 & \times .06 & = . \\
 & \times .005 & = . \\
\text{———} & \times .000\tfrac{1}{6} & = . \\
\hline
 & & .
\end{array}
$$

$$
\begin{array}{llll}
r/6 \times \ . & \times \$ & = \$ & \text{Int. Ans.} \\
\$ & + \$ & = \$ & \text{Amt. Ans.}
\end{array}
$$

Example: Find the Interest and Amount at 7% for a note for $360, dated August 27, 1905 and paid December 8, 1911.

$$
\begin{array}{ll}
1911\text{—}12\text{--}8 & 6 \times .06 = .36. \\
1905\text{—}\ 8\text{-}27 & 3 \times .005 = .015 \\
\hline
 & 11 \times .000\tfrac{1}{6} = .001\tfrac{5}{6} \\
6\text{—}\ 3\text{-}11 & \\
 & \quad .376\tfrac{5}{6}
\end{array}
$$

$7/6 \times .376\tfrac{5}{6} \times \$360 = \$158.27$ Int. Ans.

$\$158.27 + \$360 = \$518.27$ Amt. Ans.

Give much rapid oral work in the solution of problems in Interest. As the Rate at 6% for 60 days is .01, to find the Interest for 60 days, move the decimal point two places to the left; to find the interest for 30 days, move the point two places to the left and divide by 2; to find the Interest for 120 days, move the point and multiply by 2. For other terms, various combinations of the above method may be used.

The Interest on $240 for 60 days at 6% is $2.40; for 30 days, $1.20; for 90 days, $3.60; for 120 days, $4.80. To find the Interest at 5%, find 5/6 of the Interest at 6%.

XXVII. COMPOUND INTEREST.

Compound Interest, though it is illegal for general use, is still employed by saving banks, which generally compute Interest semi-annually and add this to the principal for a new principal upon which the Interest for the next term is computed. Letting R equal the Rate for the term of Interest, and t the number of terms, develop the formula below:

Example: Find the Amount of $300 compounded annually for 3 years at 4%.

Example: Find the Interest on $400 compounded semi-annually for 3 years at 4%.

$$(1.00+R)^t \times B = A$$

$$1.04^3 \times \$300 = \$347.46$$

$$1.02^6 \times \$400 = \$450.46$$

$450.46—$400=$50.46 Compound Int. Ans.

Several simple examples should be worked by actually multiplying them out, as in the first example above multiplying 1.04 by itself three times and $300 by the result. Then more difficult examples like the second should be worked by use of the following table. State the example exactly as above, then in the table in the column for 2% and opposite the year 6 will be found the Amount of $1.00 at 2% for 6 years, which is the same as the Amount of $1.00 for 3 years at 4% compounded semi-annually. Multiplying this $1.12616 by 400 gives the desired result, $450.46.

Table showing the amount of $1 at compound interest for:

Year	2 Per Cent	2½ Per Cent	3 Per Cent	3½ Per Cent	4 Per Cent
1	1.02000	1.02500	1.03000	1.03500	1.04000
2	1.04040	1.05063	1.06090	1.07123	1.08160
3	1.06121	1.07689	1.09273	1.10872	1.12486
4	1.08243	1.10381	1.12551	1.14752	1.16986
5	1.10408	1.13141	1.15927	1.18769	1.21665
6	1.12616	1.15969	1.19405	1.22926	1.26532
7	1.14869	1.18869	1.22987	1.27228	1.31593
8	1.17166	1.21840	1.26677	1.31681	1.36857
9	1.19509	1.24886	1.30477	1.36290	1.42331
10	1.21899	1.28009	1.34392	1.41060	1.48024

XXVIII. BANK DISCOUNT.

As an introduction to the subject of Bank Discount, the teacher

should explain to the class the two chief functions of banking: deposit and loan or discount. She should provide herself through some local bank with the various blank forms used in banking: the pass book, the deposit slip, the check book including checks and stubs, the bank note, and the draft.

All banks receive money on deposit for safe keeping, sometimes allowing Interest, sometimes not. Saving banks, which do not accept deposits for checking purposes, allow Interest on all sums left on deposit a stated length of time. Commercial banks allow Interest on sums left on deposit without being checked upon. When money is deposited merely for checking purposes, in some cases no Interest is allowed; in others, Interest is allowed on unchecked balances.

The teacher should further explain the method and purposes of checking; how personal accounts between individuals or firms are balanced by the use of checks instead of money. It is well to explain how such checks often pass from hand to hand, especially between banks, in the payment of debts or the settlement of accounts, before they are finally presented for payment at the bank upon which they are drawn. In this connection the function of the New York Clearing House should be explained.

Review Simple Interest and show that Bank Discount is nothing more nor less than Simple Interest paid in advance by the borrower to the bank for the use of the bank's money. Discounting notes is the bank's method of loaning. When a person of good financial standing. or one who can give good security for the repayment of the money, wishes to borrow, say $400, from the bank; he draws up a promisory note payable to himself or another at the bank in 30, 60, 90, or 120 days. The bank generally charges for this loan the legal rate of interest for the given time and collects this charge in advance. Suppose the time to be 60 days, the Interest charge, or the Bank Discount will be $4. Thus the bank loans the borrower $400 and receives in return the promisory note and $4 for the use of the loan. Or what is the same thing, the bank deducts the $4 from the $400 and pays out the balance, $396, called the Proceeds. Such examples differ from Simple Interest only in that the Interest is subtracted to find the Proceeds instead of being added to find the Amount due when the note is payable. If the borrower in the above case had wanted the use of the money for more than 120 days, instead of making out the note for a longer time, he would have it renewed when it became due and would again discount the $400 for the additional time.

Frequently business men receive from their customers in payment of accounts notes payable when the maker's crops will mature. If such notes are long time interest bearing notes, they are often deposited at a bank for collection, in which case no account is made of the notes on the books of the bank, except to give the payee credit when the notes are paid. When, however, these notes are short time

non-interest bearing notes, the payee may in case of need of ready money have them discounted at the bank at any time before they are due. In such case, the discount is computed for the number of days the note has yet to run and upon the amount that will be due when the note matures. In the solution of such problems, the use of the following blank will be found advantageous:

BANK DISCOUNT BLANK.

Date | t = | Due

Discounted

Due —— ——

/6×(×.001/6)×$ =$ B Dis. Ans. t

$ —$ =$ Pro. Ans.

When the term is 30, 60, 90, or 120 days, the appropriate rate should be placed above the parenthesis at once without multiplication.

Example: Find the Bank Discount and the Proceeds of a $120 non interest bearing note dated June 3 and due in 90 days, if discounted at the bank July 12 at 5%.

	90		June 27
Date—June 3	t = 51	Due—Sept. 1	July 31—19
			Aug. 31—31
Discounted—July 12		Due	Sept. 1— 1
			90 51 t

5/6×(51×.001/6)×$120=$.85. B. Dis. Ans.

$120—$.85 =$119.15. Pro. Ans.

Before giving problems to find the Bank Discount, it would be well to give drill in finding the date due and the term of discount. First make a quick estimate as to date due. In the above example the date due must be about Sept. 3. If in a 90 day note the date due as finally found differs more than two days from the estimated date, some error must have been made. To find the exact date, place the number of days the note is to run, 90, below the column; after "June" the number of days after June 3; after "July" and "Aug.", the number of days in those months respectively, and after "Sept." the number necessary to make the total 90. Do not add and then subtract from 90, but adding say: 9 and 1 are 10, placing the 1 in the blank opposite "Sept." Continuing say: 3—9. Thus getting the date Sept. 1.

To find the term of discount, in the second column after "July" place 19, the number of days after July 12, the date of discount. After the remaining months, July and Aug., place the same numbers as in the first column. Add and the result is the term of discount.

Before even this work can be done accurately, each child must have at tongue's end the number of days in each month.

As the Rate of Discount at 6% for one day is .001/6, to find the Discount for a given number of days, move the point three places to the left, divide by 6, and multiply by the given number of days. Find the discount on $1200 at 6% for 17 days. Mentally moving the point three places to the left and dividing by 6 gives $.20, the Discount for one day, and multiplying this by 17 gives $3.40, the Discount for 17 days.

XXIX. DISCOUNTING INTEREST BEARING NOTES.

As Discount is computed upon the sum due at maturity, the Amount of an interest bearing note instead of the face would be the Base in Bank Discount; therefore find the Amount by the 6% method and use this Amount as the Base in the Bank Discount blank.

Example: Find the Proceeds of a 90 day note for $400 bearing Interest at 5%, dated Jan. 10, 1912, and discounted Mar. 5 at 6%.

	90		Jan.	21
Date—Jan. 10 1912	t = 35	Due—Apr. 9.	Feby.	29
			Mar.	31—26
Discounted—Mar. 5		Due	Apr.	9— 9

5/6×.005×$400 =$5 I.
$5 +$400 =$405 A.
6/6×(35×.001/6)×$405 =$ 2.36 B. Dis.
$402.64 Pro, Ans.

ADDENDA.

PREVENTION OF COPYING.

1. Remove the possibility of copying by seating the class in a manner to prevent it, and by giving different examples to neighboring children at the board. On tests with all teachers and at all times with inexperienced teachers and generally in the lower grades, this method is advisable, but it does not get at the root of the evil. The tendency to copy remains and passive attention is relied upon, whereas the power of active attention with effort should be developed. To accomplish this end, the teacher must be constantly on the alert and she must employ various means to prevent copying when the possibility and temptation are always present.

2. Inflict some penalty. (a) Turn the erring child around with his back to the board till the rest of the class finish the example. (b) Mark him zero for the example. (c) Send him to his seat. But all these appeal to a low motive and the child will still copy when sure of avoiding the penalty. However he is given some discipline in that he inhibits a desire to look at his neighbor's work and thus strengthens his active attention.

3. Appeal to his sense of shame or pride or better his self respect by making him see how other people regard such actions. Here again he will continue to copy when he thinks he is sure of avoiding detection.

4. Show the child the practical results of copying, that it prevents his understanding the subject and keeps the teacher from knowing when to give him proper help and hence he will probably fail and remain in the grade another term, or that possibly he will be demoted to a lower grade. This appeal too will be of only temporary value, as when he comes to take a test, he knows that the practical result is the other way around.

5. Make the child see that copying prevents mental development, that instead of gaining mental power he is weakening his ability to think in the future. This will appeal to the ambitious child but the child who thinks only of the immediate present will receive little stimulus.

All of these fail in the end unless reinforced by some motive that applies with equal force to all possible cases. Some of these means will be found effective under some circumstances and ineffective under others. By these the child who cannot be reached by a higher appeal must be reached.

6. Appeal to the moral sense. It is not necessary to paint copying in black letters and brand it as cheating and stealing. Many children do not have sufficient moral sense to realize that it is wrong,

and the use of strong language is apt to do more harm than good. If possible, however, the child should be made to see the evil, that it is really wrong, that it is unfair to the rest of the class. How to accomplish this will depend upon the child and upon circumstances. Sometimes the appeal should be to the whole class and at times to the child privately. Many children if convinced of the wrong will cease the evil practice, while of course others will deliberately continue to copy knowing it to be wrong. With the latter some of the other means discussed will be essential.

ONLY ONE DIFFICULTY AT A TIME.

Under the discussion regarding teaching a new operation, emphasis was laid upon introducing but one new difficulty at a time. A few specific cases will here be given.

In **column addition**, which should be begun as soon as the child knows the facts sum seven, the new difficulty is remembering the sum of the first two digits and adding this to the third digit, with but one of the two numbers in sight. As a preparation for this, it might be well to place some number, as 2, on the board and then have the class add to this any number from 1 to 5 dictated by the teacher. Suppose the column to be 2-2-3, ask the child the sum of the first two digits, then ask him the sum of this number and the last digit. Focus the child's mind upon the thought that he is to add the sum of the first two digits to the third, thus making the new idea as vivid as possible, the strong first impression. At first the work will have to be oral, the teacher questioning and the child answering, then the child doing all the oral work alone, and finally adding mentally and giving the final answer only.

The next difficulty in column addition is carrying and here again the child's mind must be focussed upon whether there is a carry and that he must add it as soon as he determines that there is. Of course if any of the addition facts used are not thoroughly known, the new difficulty will not be the only one that the child must contend with, the attention will be distributed, and hence the first impression will not be as vivid as it would be if the attention could be focussed upon the one new idea. This emphasizes the importance of reducing the number facts to habit.

The third difficulty is adding a column containing facts not included among the forty-five elementary number facts. This has been fully discussed under series and column addition, but under each new class of series the teacher must emphasize and hold the child's attention upon the new idea till it is fully grasped and can be applied without hesitation. As soon as this point is reached and no sooner, the child is ready for another difficulty. It is the piling up of difficulty after difficulty without mastering each in turn that creates havoc in arithmetic teaching.

In subtraction the first difficulty is the idea of subtraction itself. As already explained, this idea should be brought out by use of simple problems within the child's own experience and comprehension. Do not introduce the Austrian method till sure that the subtraction idea is fully grasped. So too the preparation for subtraction must be fully in hand before formal subtraction begins. Then these two ideas must be combined; the child finds the difference by applying the preparation. If very simple examples are first used, he will see the connection between the two ideas. Next is the carrying, now used in place of the old borrowing. The method of focussing upon this new difficulty has been fully discussed.

As in addition and subtraction, so in multiplication and division, focus the child's attention closely upon each new idea in turn. The steps have already been indicated, in some cases merely by progressive type examples. These type examples should be closely studied by the teacher to determine what is the essential new difficulty introduced in each. Particularly note the general class of long division examples in which the quotient figure cannot be directly found by dividing the first digit or the first two digits of the dividend by the first digit of the divisor. The new difficulty upon which to focus here is that the subtrahend cannot be larger than the minuend. To be sure, this difficulty may have come up incidentally before, due to a child's carelessness, but then the teacher should merely have called attention to the particular error, and the special difficulty should have been left to a later time, as then some other idea was under focus. Now the teacher places on the board several examples introducing the new difficulty and questions the class regarding each step, and then when the difficulty is reached in the example, she calls attention to it and asks the class to watch hereafter for this trouble. After the first example, the class should see the error as they are looking for just this. After several examples have been worked at the board by one child under the direction of the class and the supervision of the teacher, the whole class should be sent to the board and an example dictated with special instructions to watch for the new difficulty. Until the class can work such examples without effort, the teacher will find it necessary to remind the different individuals that they are watching for something.

Similarly handle the principle that the remainder must be smaller than the divisor.

In teaching reading and writing numbers above 1000, focus on the fact that the digits between two commas or to the left of a comma should be read as if they stood alone and then the name given as thousand or million as the case may be. Then as a preparation for the next step, focus on the fact that there must be three figures between any two commas or to the right of a comma. Emphasize this. calling attention to it repeatedly.

Then in reading numbers in which there are no hundreds or no hundreds and tens, focus on the fact that the digits or digit are placed to the right of the period and that the vacant places are filled in with ciphers in order to make three places in the period.

To teach writing such numbers, the teacher should write on the board 25,067 and have the class at the board copy this, then have them write dictated numbers directly below it in the proper place. In dictating 27,048, the teacher should say "27 thousand" and pause long enough for the class to write 27 and place the comma after it. Always hesitate thus and insist on the placing of the comma at once. As soon as the class has written the comma, continue the dictation—"48", letting the voice fall for the first time to indicate that the dictation is complete, and question as to where the digits must be placed. Ans. To the right of the period. Under what figures? Ans. Under 67. How many figures must there be in a period? Ans. Three. Then what must you do to make three places? Ans. Write in a naught.

Similarly teach numbers in which two naughts must be inserted. After a time drop the questions but caution the class to be on the watch for the new point just learned. If some member of the class still has trouble and fails to insert the naught, question him as before.

In all work, fractions, denominate numbers, decimals, percentage, proportion, and square root, follow the same general plan of focussing closely for some time upon the specific difficulty introduced in each type example. In the end time will thus be saved.

In **Proportion** the order of steps would be first a preparation showing that if the product of two factors equals the product of two other factors, and one of the factors is missing, it may be found by dividing the product of one pair of factors by the factor in the pair from which the one is missing. Place the two factors to be multiplied above the line and the single factor below the line and cancel.

Introduce proportion by means of diagrams illustrating the fact that at a given time objects of different heights cast shadows of different lengths; the higher the object, the longer the shadow cast. Also use any other illustrations that will make the subject clear.

Next teach the statement of the proportion, emphasizing that like numbers should be grouped in one ratio. The heights of the objects should be placed in one ratio and the lengths of the shadows in the other. Give much practice in stating problems before attempting solution.

Follow this with the principle that the product of the means equals the product of the extremes, again giving practice in determining what terms should be multiplied together. Then basing upon the preparation mentioned above, teach the method of finding the required term.

The next difficulty will be inverse proportion. Use a simple example as: If one boy piles a certain amount of wood in 3 hours, how long will it take two boys working at the same rate to pile the same amount? Have the like numbers stated in one ratio, then place the odd term in either the mean or the extreme in order that the result may be larger or smaller than the given odd term as desired. That is, if the result should be larger than the odd term, place the odd term so that it will multiply the larger of the other two terms; if the desired result should be smaller than the odd term, place the odd term to multiply the smaller of the two other terms. Hereafter the first thing to be considered in a problem is, whether in the new condition the result should be greater or smaller than in the first condition stated, then the example should be so stated that the result will be greater or smaller as desired.

If compound proportion is to be taught, first put down the ratio containing the unknown as a mean; then state each condition so that the unknown will be increased or decreased as the condition requires. Note that in determining how to state each ratio, its effect upon the unknown alone is considered and all other conditions are ignored for the time being. If the unknown will be increased by a given condition, place the larger term in the extreme and the smaller in the mean; if the unknown will be decreased by the condition, place the smaller term in the extreme.

Example: If it takes 50 men 12 days of 10 hours each to dig a ditch 80 rods long, 6 feet wide, and 4 feet deep; how many men working 15 days of 8 hours each will be required to dig a similar ditch 96 rods long, 5 feet wide, and 6 feet deep?

Will it take more or less men working 15 days than it takes when they work 12 days? Ans. Less. Then write the smaller nubber, 12, in th extreme to multiply the 50, and the 15 in the mean to divide the 12×50.

Will it take more or less men if they work 8 hours a day instead of 10 hours? Ans. More. Then state so that the result will be more; that is, multiply the extreme, 50, by the larger number 10, by placing it in the other extreme. Where shall we place the 8?

Will it take more or less men to dig a ditch 96 rods long than to dig one 80 rods long? Then place to obtain more.

Will a 5 foot ditch require more or less men than a 6 foot ditch? Etc.

$$\begin{array}{ccccccc} & & & & 15 & : & 12 \\ & & & & 8 & : & 10 \\ 50 & : & \times & = & 80 & : & 96 \\ & & & & 6 & : & 5 \\ & & & & 4 & : & 6 \end{array}$$

Place all the extremes above the line and all the means below and cancel.

The following progressive steps in square root should be emphasized in turn:

1. Mentally extract the square root of such numbers as 16; 49; 81; 100; 144; 400; etc.

2. Mentally extract the square root of the largest square contained in any number of one or two digits. The square root of the largest square in 85 is 9; that is the largest square in 85 is 81, whose root is 9. The square root of the largest square in 80 is 8. Etc.

3. Find the square root by factoring. The square root of 36 equals the square root of $4 \times 9 = 2 \times 3 = 6$. The square root of $225 =$ the square root of $9 \times 25 = 3 \times 5 = 15$. Or the square root of $225 =$ the square root of $3 \times 3 \times 5 \times 5 = 3 \times 5 = 15$. Therefore to find the square root of a perfect square, find its prime factors and take one factor from each pair of identical factors as the factors of the root. Find the square root of 3969. $3969 = (7 \times 7) \times (3 \times 3) \times (3 \times 3)$, therefore the square root of 3969 is $7 \times 3 \times 3$, or 63.

4. Teach pointing off into periods of two figures each both ways from the decimal point. 5'67.57'80; 5'83'82.60'00'00.

5. Develop the regular method of extracting the square root of integers of three or four digits. Teach by means of diagram or rule. See Stamper, op. cit., pp. 116-118. After developing method of solution, use "chalk and talk" method with class at board.

6. Square root of numbers of three periods.

7. Exact root of integer and decimal of two and four decimal places.

8. Appropriate root found by annexing ciphers to the decimal.

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Experiments in the first grade along the line suggested in Myers, Experimental Psychology, pp. 72-90, indicate that a class will learn a set of facts logically grouped together as soon as, if not sooner than, it will learn a single fact by itself. For example, to teach the facts whose sum is six, have each child take a six inch rod and find all the pairs of rods that placed together will be equal in length to the six inch rod. Have them write these facts in the four ways and give various drills. The next day have them write or tell all the facts in the "six family" that they can remember. Then have them find any forgotten facts by means of the rods. Before taking up the "seven family", see that they thoroughly know all previous "families". Children enjoy this work as it appeals to the puzzle instinct.

ERRATA.

Page 33. $\frac{124}{6/0)744/0}$ $\frac{8}{8/000)65/721}$—1721

Page 49, above Section XV, inverted figure 7 (7) should be a cipher.

Page 41, fourth line from bottom. **the.**

Page 42, line 11. **reduced.**

Page 50, XVIII. MULTIPLICATION.

Page 57, line 4. 5160 should be 51600.
XIV. line 10. 11.52 should be 115.2.

Page 61, See that each .17 in (b) is directly below the corresponding .17 in (a).

Page 71, line 5 from bottom. **money.**

Page 77, line 4 from bottom, .005 should be .015. Also 90 should be below left hand column of dates and 35 below right hand column.

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